A Model of Raiffa's Bargaining Solution for a Strategic Game

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Abstract

Raiffa's solution to the bargaining problem, outlined in Luce and Raiffa (1957), is the point where the *negotiation curve* - a sequence of points that constitute step-by-step improvements from the status quo in the feasible payoff space - meets (possibly in the limit) the efficient boundary of the feasible region. This paper is devoted to clarifying the logic of Raiffa's solution and its relationship with the Nash bargaining solution.

JEL Classification: C71, C78
Keywords: bargaining, Nash bargaining solution, Raiffa's bargaining solution, negotiation curve

1 Introduction

We consider bilateral bargaining in a two-player strategic form game, where a randomly chosen player proposes a correlated strategy for playing the game and the other accept or rejects. We describe the bargaining in terms of the two oldest solution concepts in that field, those of Nash (1950, 1953) and Raiffa (1953). There are extensive form bargaining models that give these solutions in some limiting allocation, namely those of Binmore et al. (1986) for Nash's solution; and, Myerson (1991) and Sjöström (1991) for Raiffa's solution. In all these extensive forms, bargaining ends in the first stage in subgame perfect equilibrium.

While Nash proposed a single-stage demand game as strategic support for his solution, Raiffa's main conceptual innovation is the *negotiation curve*, which starts at the status quo point and moves to the Pareto frontier, presumably in real time. This suggests that

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Raiffa conceived of bargaining as a process of initial disagreement and delay. In more recent models of bargaining, delay is usually a function of asymmetric information or, at least, of uncertainty. One approach to modeling the process is taken by Hu and Rocheteau (2020) and Rocheteau et al. (2021), who break up the negotiation into several stages, and the outcome in a stage becomes the status quo point for the next. We adopt a different approach, considering a finitely repeated game in which payoffs are per period and an agreement is only valid for the period in which it is obtained.

Raiffa's solution to the bargaining problem, described in Raiffa (1953) and outlined in Luce and Raiffa (1957)[1], is the final payoff point on the efficient boundary, in a sequence of points starting from the status quo, that constitute step-by-step improvements in players' payoff positions. Geometrically, the Raiffa sequence of payoff points may be seen as a potentially non-linear negotiation curve in the feasible payoff space that eventually meets the efficient boundary, possibly in the limit. It is natural to be interested in the relationship between Raiffa's solution and Nash's solution, especially given the apparently different ideas underlying them. Luce and Raiffa, in their book, make the following remarks:

"... in the continuous motion model, the slopes of the negotiation curve and of the Pareto optimal curve are of the same magnitude at their point of intersection, but of the opposite sign. If one "linearizes" this model by demanding that the negotiation curve be a straight line having this same relation between its slope and that of the Pareto optimal curve at their point of intersection, then the arbitrated point is Nash's point where the product is a maximum. " Luce and Raiffa (1957)

The relationship between Raiffa's solution and Nash's solution is the central concern of this paper. We show that as bargaining frictions vanish (to be made precise later), Raiffa's solution converges to Nash's solution. As one eminent reader of a draft of this paper commented, one could think of this enterprise as an exploration of scientific history.

The strategic aspects of Raiffa's solution were studied only much later. Sjöström (1991) shows that the equilibrium payoffs of a finite horizon bargaining model approximate Raiffa's solution as the offers become more frequent. Myerson (1991)[2] describes a finite horizon bargaining model, which in the limit as the horizon becomes infinite, implements Raiffa's solution as the unique subgame perfect equilibrium value. These models, however, do not reveal any relationship between Raiffa's solution and Nash's solution. In section 5, we use a version of Myerson's bargaining model that has discounting frictions to define a *noncooperative* Raiffa solution as its equilibrium bargaining value. Theorem 1 is our first result of the

[1] pp. 136-137 of the Dover publications 1989 reprint
[2] pp. 393

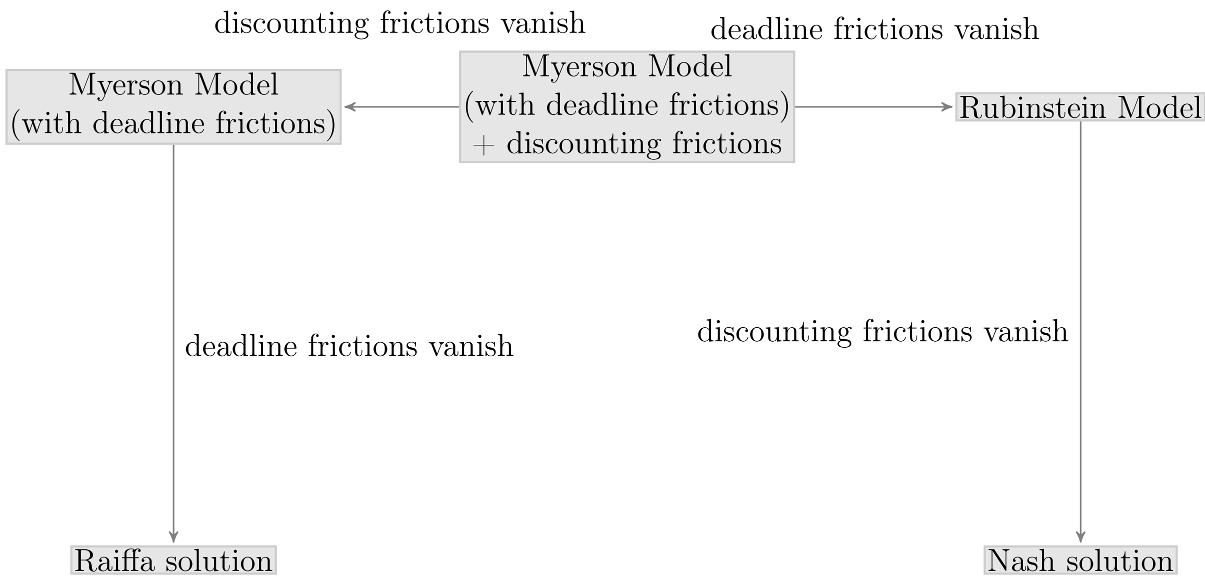

Figure 1: Relationship between the Raiffa solution and the Nash solution via a version of Myerson's bargaining model.

paper which establishes that as the bargaining frictions vanish, the noncooperative Raiffa solution so defined converges to the Nash bargaining solution with respect to the *deadlock point*, $\mathbf{0}$, for the model. The reason behind this result is revealed by looking at the behavior of the model as we change the order in which the frictions disappear; this is illustrated in Figure 1. On letting the discounting frictions disappear first, we are back to Myerson's bargaining model which leads to Raiffa's solution when deadline frictions disappear. However, on letting the deadline frictions disappear first, we get Rubinstein's bargaining model for the strategic form game with $\mathbf{0}$ as the deadlock point. We know from Binmore et al. (1986) that as the discounting frictions disappear next, we get Nash's solution with respect to $\mathbf{0}$. The crucial argument in the proof is to establish the validity of the limit interchange. This key technical property is proved in Lemma 10.

The modified Myerson's model is deficient, however, in two respects. As Rubinstein's bargaining model has $\mathbf{0}$ as the deadlock point, Theorem 1 is a limited result in that the convergence-to-Nash result does not obtain for an arbitrary status quo point. Moreover, bargaining stops in the first period. As a way of addressing these limitations, in section 7, we study a bargaining model for a finitely repeated game with an exogenous status quo; and use it as the basis for once again defining a *noncooperative* Raiffa solution, but with respect to this model. The crucial difference with Myerson's bargaining model for a one-shot game is that the prevailing contract at the current date determines the contract at the next date by determining the continuation value of rejection for the responder. This makes it easier to conceive the current period payoff as constituting the status quo for the next period. The proposer in the current period then offers the next date's contract so as to make the responder indifferent between accepting and rejecting the offer. Theorem 2, our central result, establishes that as the deadline and then the discounting frictions disappear, the noncooperative Raiffa solution with respect to this model with the pre-specified status quo converges to the Nash bargaining solution with respect to the same status quo.

The bargaining model of section 7, we think, is better suited as a model of Raiffa's solution because unlike the models of Myerson and Sjöström, it features an equilibrium path[3] of negotiated contracts, which could be conceived as a nontrivial[4] *noncooperative* negotiation curve - an idea central to Raiffa's conception of the bargaining process. The noncooperative negotiation curve is in contrast to the (cooperative) Raiffa negotiation curve discussed in section 4. Nevertheless, the Raiffa negotiation curve does show up in the model as a sequence of threat points, recursively computed using the backward induction algorithm. With discounting frictions present, players care about the payoffs they realize all along the noncooperative negotiation curve. However, as the model becomes frictionless, players only care about where they finally end up on the curve. The final point is the Nash bargaining solution because as the deadline frictions disappear, the model becomes the random-offer Rubinstein style bargaining model for the infinitely repeated game with an exogenous status quo, which is studied in section 6. We know from Proposition 2 in that model that as the discounting frictions disappear, the solution converges to Nash bargaining solution with respect to the given status quo.

The plan of the paper is as follows. In section 2, we discuss closely related literature. Section 3 develops some basic geometry of the polytope formed by strategic game payoffs. Raiffa's bargaining solution for a strategic game is discussed in section 4. In section 5, we analyze a finite horizon bargaining model for a one-shot play of the game. Section 6 is an interlude to the main theme, where we analyze a random offer Rubinstein style bargaining model for an infinitely repeated game with a status quo, primarily to get a convergence-to-Nash result. However, in Appendix C, we also use this model to place Raiffa's mediation interpretation of his solution in a strategic setting. Specifically, we add a mediator to the model who makes an offer to both players every period - an offer that reflects the value of strategic bargaining and is therefore accepted by both players every period. It is on this mediated path that we see the Raiffa negotiation curve eventually leading up to the Nash solution. In section 7, we then develop and analyze a finitely repeated game with a status quo. The analysis in the models developed in the main body of the paper appeal to results that parallel Binmore

[3] stochastic in the present model

[4] typically consisting of multiple steps

et al. (1986), although in the context of a strategic form game. For this reason, we lay out the risk-of-breakdown model in Appendix A and the time preference model in Appendix B. Finally, we conclude in Section 8.

2 Related Literature

We follow Nash (1953) in taking a strategic form game as the primitive environment for bargaining. Unlike Nash however, players use a sequential strategic bargaining process, in the style of Rubinstein (1982), Binmore and Dasgupta (1987), and Binmore et al. (1986), to write a contract. We think that taking a strategic game as the basic data for bargaining theory is a desirable goal, as it allows us to compare noncooperative and contractual modes of behavior in the same unified language. Raiffa also takes the strategic form game as the primitive when discussing his solution in Luce and Raiffa (1957).

Table 1 contrasts some well known bargaining solutions like the Nash solution and the Rubinstein solution with the (cooperative) Raiffa solution and the noncooperative Raiffa solution defined in this paper. The Rubinstein solution has the property that negotiation cost (arising from impatience) off the equilibrium path leads to zero negotiation costs on the equilibrium path. In contrast, the Raiffa solution has the property that it traces out an actual path from the status quo to the solution.

As far as we are aware, Sjöström (1991) and Myerson (1991) were among the first to develop the noncooperative foundations of Raiffa's bargaining solution. Rocheteau et al. (2021), in recent work, formalize decentralized trade of a divisible asset (valued linearly) as a gradual bargaining problem (consisting of time-indexed Pareto-frontiers with an initial status quo) in the sense of O'Neill et al. (2004). They construct strategic bargaining models with the feature that the owner of the asset makes the asset incrementally available for negotiation. This leads to step-by-step negotiations in the model and as the horizon becomes infinite, the equilibrium payoffs, described by a system of differential equations, converge to the ordinal solution of O'Neill et al. (2004). In a companion paper, Hu and Rocheteau (2020), by taking a model with the feature that the owner of the consumption good makes it incrementally available for negotiation, show that the solution converges to the proportional bargaining solution of Kalai (1977).

In the cooperative game theory literature, Raiffa's solution (discrete version) has been axiomatized by Anbarci and Sun (2013) and further by Trockel (2015) using axioms related to Kalai (1977)'s axiom of step-by-step negotiation. In particular, Trockel (2015) axiomatizes Raiffa's solution as the unique solution satisfying efficiency, scale covariance, symmetry and

	Nash Solution	Raiffa Solution	Rubinstein Solution	Noncooperative Raiffa Solution
Nature	axiomatic	descriptive with noncooperative flavor	strategic	strategic
Description	unique point satisfying certain axioms	unique point where negotiation curve meets efficient boundary	unique SPE of a strategic bargaining model with discounting frictions	unique SPE of a strategic bargaining model with per-period payoffs; with deadline and discounting frictions
Efficiency	efficient	efficient (assuming negotiation takes no time)	efficient	possibly inefficient with frictions; convergence to efficiency
Condition under which same	-	when efficient boundary is linear	same as time-preference Nash solution with $\mathbf{0}$ as deadlock point	same as time-preference Nash solution with respect to deadlock point when frictions vanish
Negotiation curve	none	possibly nonlinear and explicitly defined	straight line (one step) from deadlock point to solution	stochastic equilibrium path of prevailing contracts; straight line (one step) from deadlock point to Nash solution in the limit; Raiffa's negotiation curve appears as backward-in-time sequence of threat points

Table 1: Comparison of bargaining solutions

a consistency property by which the midpoint solution of TU bargaining games can be extended to NTU bargaining games. A generalized version of Raiffa's solution is axiomatized by Diskin et al. (2011).

3 Strategic Game and its Geometry

The payoff environment for bargaining is described by a strategic form game between two players indexed by $i \in N = \{1, 2\}$. Player i's feasible (pure) actions a_i lie in a finite set $\mathcal{A}_i$; and mixed actions in the probability simplex $\Delta(\mathcal{A}_i)$. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be the finite set of feasible pure action profiles $\mathbf{a} = (a_1, a_2)$. Let player i's preferences over $\Delta(\mathcal{A}_i)$ be represented by the expectation of a stage game vNM payoff function given by $u_i : \mathcal{A} \mapsto \mathbb{R}$ and let $\mathbf{u} = (u_1, u_2)$ be a profile of vNM payoff functions. The stage game G is defined by the components $\langle N, \mathcal{A}, \mathbf{u} \rangle$ and is viewed as the underlying payoff environment in which players get opportunities to propose contracts and also to accept or reject contracts that are directed to them.

Let $u(\mathcal{A})$ be the image of $\mathcal{A}$ under u. Then $u(\mathcal{A})$ is a finite set in $\mathbb{R}^n$. Let $\mathcal{F} := co(u(\mathcal{A}))$ be the convex hull of the set $u(\mathcal{A})$. Then $\mathcal{F}$ is a polytope in $\mathbb{R}^n$. Let $\partial\mathcal{F}$ be the (strongly) efficient boundary of $\mathcal{F}$. That is, $\partial\mathcal{F} = \{\mathbf{u} \in \mathcal{F} : \nexists \mathbf{u}' \in \mathcal{F} \text{ such that } \mathbf{u}' \geqslant \mathbf{u} \text{ and } \mathbf{u}' \neq \mathbf{u}\}$. Let $\partial^-\mathcal{F}$ be the weakly efficient boundary of $\mathcal{F}$. That is, $\partial^-\mathcal{F} = \{\mathbf{u} \in \mathcal{F} : \nexists \mathbf{u}' \in \mathcal{F} \text{ such that } \mathbf{u}' \gg \mathbf{u}\}$. The efficient boundary is contained in the weakly efficient boundary. That is, $\partial\mathcal{F} \subseteq \partial^-\mathcal{F}$. For any compact set $S \subseteq \mathbb{R}^n$, let $m_i(S) := \min_{\mathbf{u} \in S} u_i$ and let $M_i(S) := \max_{\mathbf{u} \in S} u_i$. If $S \subseteq \mathcal{F}$, then these numbers are interpreted as the minimum and the maximum payoff respectively, of player i, among the payoff vectors in the set S. We record two elementary properties about these numbers.

Lemma 1. *For any player $i = 1, 2$, we have $M_i(\partial\mathcal{F}) = M_i(\mathcal{F})$.*

Proof. We prove the claim for player 1. The proof for player 2 is similar. As $\partial\mathcal{F} \subseteq \mathcal{F}$, we have $M_1(\partial\mathcal{F}) \leqslant M_1(\mathcal{F})$. To establish the reverse inequality, take any point $(M_1(\mathcal{F}), u_2) \in \mathcal{F}$. Then it follows from the definition of $\partial^-\mathcal{F}$ that $(M_1(\mathcal{F}), u_2) \in \partial^-\mathcal{F}$. As $\partial^-\mathcal{F}$ is a compact set, let $u_2^\star := \max_{(M_1(\mathcal{F}), u_2) \in \partial^-\mathcal{F}} u_2$. Then $(M_1(\mathcal{F}), u_2^\star) \in \partial\mathcal{F}$. Therefore, $M_1(\partial\mathcal{F}) \geqslant M_1(\mathcal{F})$. This establish the lemma for player 1. Q.E.D.

Lemma 2. *The payoff vectors $(m_1(\partial\mathcal{F}), M_2(\partial\mathcal{F}))$ and $(M_1(\partial\mathcal{F}), m_2(\partial\mathcal{F}))$ are extreme points of $\mathcal{F}$.*

Proof. We prove the claim for the first point. The proof for the second point is similar. Arguing by way of contradiction, suppose there exist two distinct points (v_1, v_2) and (w_1, w_2), both in $\mathcal{F}$, and $\alpha \in (0, 1)$ such that $(m_1(\partial\mathcal{F}), M_2(\partial\mathcal{F})) = \alpha(v_1, v_2) + (1 - \alpha)(w_1, w_2)$. This implies that $M_2(\partial\mathcal{F}) = \alpha v_2 + (1 - \alpha)w_2$. Therefore, either $v_2 > M_2(\partial\mathcal{F})$ or $w_2 > M_2(\partial\mathcal{F})$. Without loss of generality, suppose it is the former. Then by Lemma 1, we have $v_2 > M_2(\mathcal{F})$. But this contradicts the definition of $M_2(\mathcal{F})$, thereby establishing the lemma for the point $(m_1(\partial\mathcal{F}), M_2(\partial\mathcal{F}))$. Q.E.D.

Let $\mathcal{C}$ be a coordinate system with $(m_1(\partial\mathcal{F}), m_2(\partial\mathcal{F}))$ as the origin. Refer to the horizontal axis as $\mathcal{C}_1$ and the vertical axes as $\mathcal{C}_2$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ divide the plane into four quadrants $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$.

Let $h_1 : [m_2(\mathcal{F}), M_2(\mathcal{F})] \mapsto \mathbb{R}$ and $h_2 : [m_1(\mathcal{F}), M_1(\mathcal{F})] \mapsto \mathbb{R}$ be functions defined by

$$h_2(d_1) := \max\{u_2 : (u_1, u_2) \in \mathcal{F} \text{ and } u_1 \geqslant d_1\}$$
$$h_1(d_2) := \max\{u_1 : (u_1, u_2) \in \mathcal{F} \text{ and } u_2 \geqslant d_2\}$$

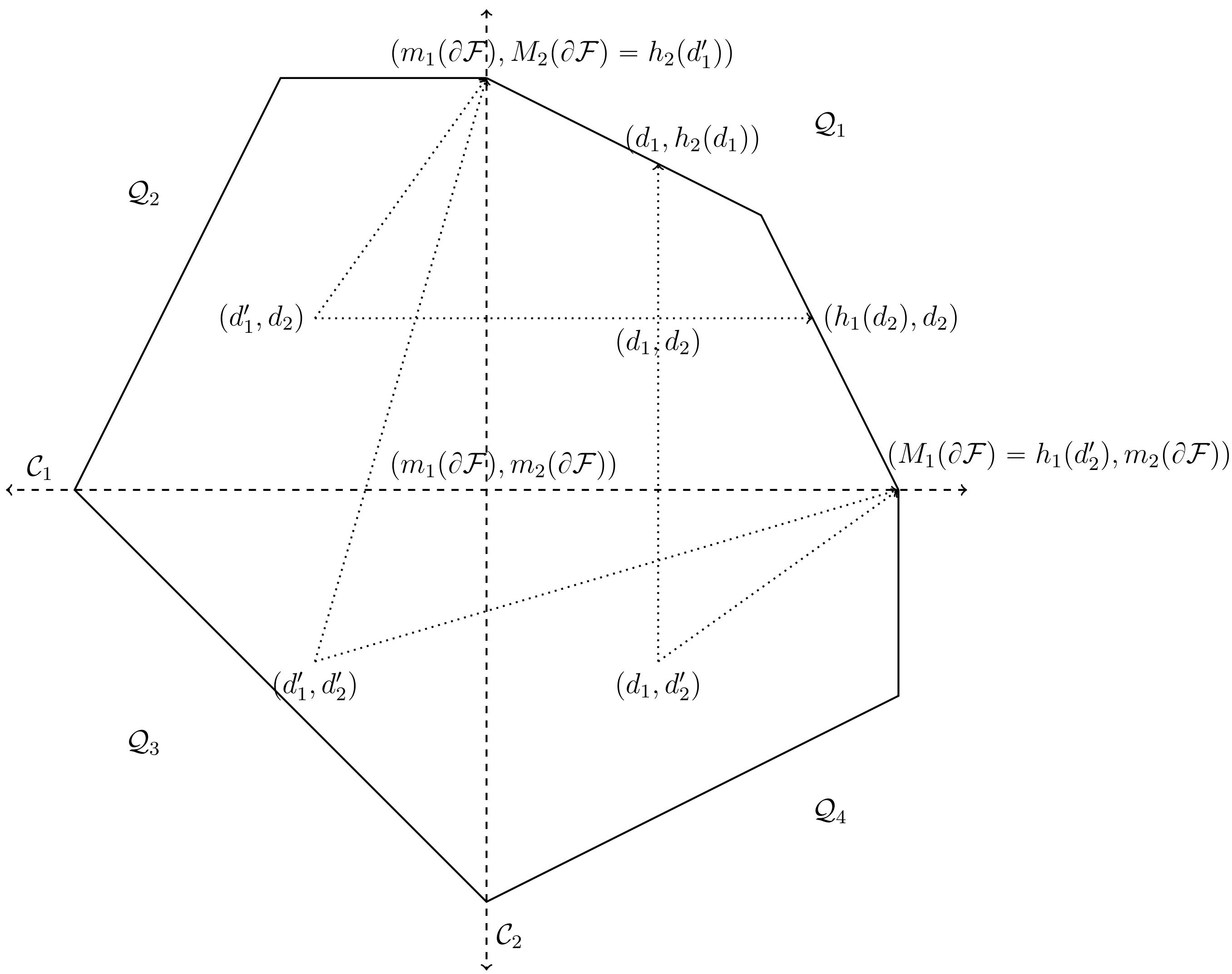

Figure 2: Geometry of $\mathcal{F} := co(u(\mathcal{A}))$ and mappings h_1 and h_2

In the maximization problems that define h_1 and h_2, the feasible sets are compact subsets of $\mathbb{R}^2$ and the objective functions are continuous. Therefore, Weierstrass' Theorem guarantees that h_1 and h_2 are well defined.

Then we have

$$h_2(d_1) = \begin{cases} M_2(\partial\mathcal{F}) & \text{if } m_1(\mathcal{F}) \leqslant d_1 < m_1(\partial\mathcal{F}) \\ \max\{u_2 : (u_1, u_2) \in \mathcal{F} \text{ and } u_1 = d_1\} & \text{if } m_1(\partial\mathcal{F}) \leqslant d_1 \leqslant M_1(\mathcal{F}) \end{cases}$$

$$h_1(d_2) := \begin{cases} M_1(\partial\mathcal{F}) & \text{if } m_2(\mathcal{F}) \leqslant d_2 < m_2(\partial\mathcal{F}) \\ \max\{u_1 : (u_1, u_2) \in \mathcal{F} \text{ and } u_2 = d_2\} & \text{if } m_2(\partial\mathcal{F}) \leqslant d_2 \leqslant M_2(\mathcal{F}) \end{cases}$$

The range of h_2 is $[m_2(\mathcal{F}), M_2(\mathcal{F})]$, and the range of h_1 is $[m_1(\mathcal{F}), M_1(\mathcal{F})]$. Figure 2 illustrates the geometry of the mappings h_1 and h_2 for an example of a strategic game polytope. Lemmata 3, 4 and 5, which follow next, are easy consequences of the definition of h_1 and h_2.

Lemma 3. *For any point $\mathbf{d} = (d_1, d_2) \in \mathcal{F}$, we have $h_1(d_2) \geqslant d_1$ and $h_2(d_1) \geqslant d_2$. If $\mathbf{d} \notin \partial^-\mathcal{F}$, then the inequalities are strict.*

Lemma 4. *The functions h_1 and h_2 are continuous, decreasing and concave on their respective domains.*

Lemma 5. *When the domain of h_1 is restricted to $[m_2(\partial\mathcal{F}), M_2(\mathcal{F})]$ and the domain of h_2 is restricted to $[m_1(\partial\mathcal{F}), M_1(\mathcal{F})]$, then h_1 and h_2 are strictly decreasing and strictly concave. Moreover, they are inverses to each other i.e. $h_2 = h_1^{-1}$ and $h_1 = h_2^{-1}$.*

Lemma 6. *For any point $\mathbf{d} = (d_1, d_2) \in \mathcal{F}$, $\Big(h_1(d_2), \max(d_2, m_2(\partial\mathcal{F}))\Big)$ and $\Big(\max(d_1, m_1(\partial\mathcal{F})), h_2(d_1)\Big)$ are points on the efficient boundary $\partial\mathcal{F}$.*

Proof. Consider the first point in the statement of the lemma. Suppose $d_2 > m_2(\partial\mathcal{F})$. This happens when $\mathbf{d} \in \mathcal{Q}_1 \cap \mathcal{F}$ or $\mathbf{d} \in \mathcal{Q}_2 \cap \mathcal{F}$. Then the first point, $(h_1(d_2), d_2)$ is a point where the horizontal line through d intersects the efficient boundary $\partial\mathcal{F}$. The opposite inequality, $d_2 \leqslant m_2(\partial\mathcal{F})$ is true when $\mathbf{d} \in \mathcal{Q}_3 \cap \mathcal{F}$ or $\mathbf{d} \in \mathcal{Q}_4 \cap \mathcal{F}$. In this case, the first point $(h_1(d_2), m_2(\partial\mathcal{F})) = (M_1(\partial\mathcal{F}), m_2(\partial\mathcal{F})) \in \partial\mathcal{F}$.

Consider the second point. Suppose $d_1 > m_1(\partial\mathcal{F})$. This happens when $\mathbf{d} \in \mathcal{Q}_1 \cap \mathcal{F}$ or $\mathbf{d} \in \mathcal{Q}_4 \cap \mathcal{F}$. Then the second point, $(d_1, h_2(d_1))$ is a point where the vertical line through d intersects the efficient boundary $\partial\mathcal{F}$. The opposite inequality, $d_1 \leqslant m_1(\partial\mathcal{F})$ is true when $\mathbf{d} \in \mathcal{Q}_2 \cap \mathcal{F}$ or $\mathbf{d} \in \mathcal{Q}_3 \cap \mathcal{F}$. In this case, the second point $(m_1(\partial\mathcal{F}), h_2(d_1)) = (m_1(\partial\mathcal{F}), M_2(\partial\mathcal{F})) \in \partial\mathcal{F}$. Q.E.D.

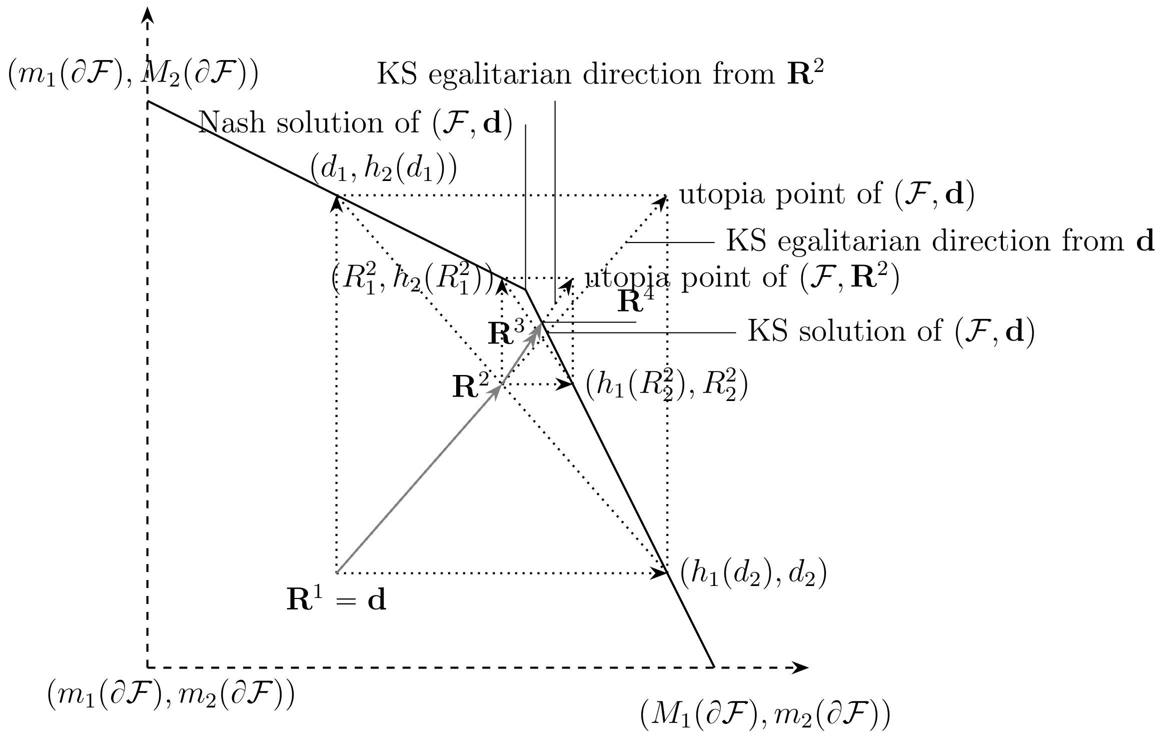

Figure 3: A finite-step piecewise-linear Raiffa negotiation curve. KS stands for Kalai Smorodinsky.

4 Raiffa's bargaining solution

Luce and Raiffa (1957) describe what is generally understood as Raiffa's sequential bargaining solution $\mathbf{R}(\mathcal{F}, \mathbf{d})$ for the bargaining problem $(\mathcal{F}, \mathbf{d})$. Myerson (1991), on pp. 393 of his book, describes it formally as the limit as $k \to \infty$ of a recursively defined sequence $(\mathbf{R}^k)_{k \geqslant 1}$ of feasible payoffs where

$$\mathbf{R}^1 = \mathbf{d}$$

$$\mathbf{R}^{k+1} = \frac{(h_1(R_2^k), R_2^k) + (R_1^k, h_2(R_1^k))}{2}$$

Figure 3 depicts the payoff vectors in the *Raiffa* sequence $(\mathbf{R}^k)_{k \geqslant 1}$. Following Raiffa's intuitive description of this sequence as a negotiation curve comprising of step-by-step improvements upon the current status quo point, we may think of the negotiation curve $(\mathbf{R}^1, \mathbf{R}^2, \mathbf{R}^3, \mathbf{R}^4)$ in three steps - first, as a movement from $\mathbf{R}^1$ to $\mathbf{R}^2$; second, as a movement from $\mathbf{R}^2$ to $\mathbf{R}^3$, and finally as a movement from $\mathbf{R}^3$ to $\mathbf{R}^4$. Figure 3 shows this piecewise linear negotiation curve in gray directed arrows. For the first step, Raiffa defines $\mathbf{R}^2$ as a fair compromise of players' most selfish offers. Geometrically, this is defined as the midpoint of the most selfish payoff offers $(h_1(d_2), d_2)$ and $(d_1, h_2(d_1))$. However, it is easily seen by

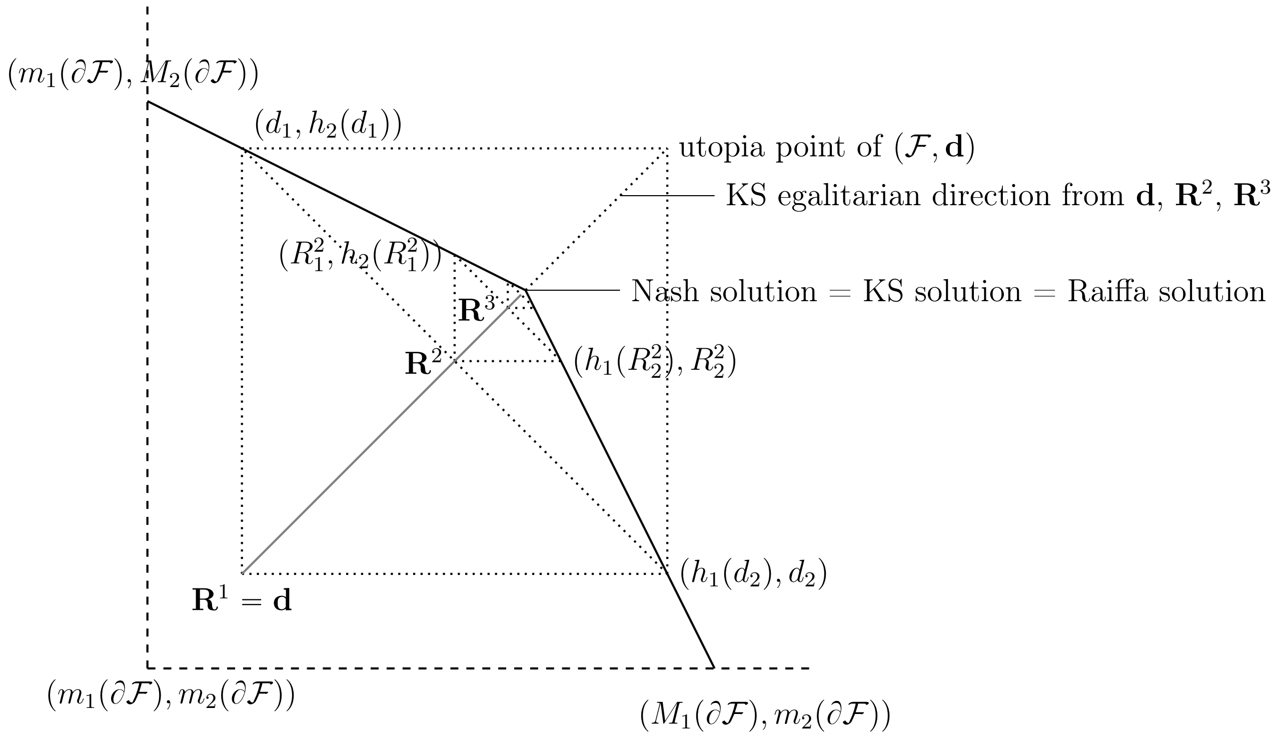

Figure 4: An infinite-step linear Raiffa negotiation curve.

completing the rectangle that the movement in the first step from $\mathbf{R}^1 = \mathbf{d}$ to $\mathbf{R}^2$ is in the direction of the utopia point of $(\mathcal{F}, \mathbf{d})$ which, following Kalai and Smorodinsky (1975), may be termed the Kalai Smorodinsky egalitarian direction from $\mathbf{d}$. The extent of movement is precisely till the point where the two diagonals of the rectangle meet. For the second step, $\mathbf{R}^2$ constitutes the status quo; and again the movement from $\mathbf{R}^2$ to $\mathbf{R}^3$ is in the Kalai Smorodinsky direction from the current status quo $\mathbf{R}^2$. For the third and final step, $\mathbf{R}^3$ is the status quo; and the movement is to $\mathbf{R}^4$ which lies on the efficient boundary and is therefore the Raiffa bargaining solution for the game shown.

In general, the Raiffa sequence may be an infinite sequence, in which case, the iterative limit of the Raiffa sequence is a point on the efficient boundary $\partial\mathcal{F}$ of the strategic game, which is considered as the Raiffa bargaining solution. Figure 4 depicts the first three payoff vectors in the (infinite) *Raiffa* sequence $(\mathbf{R}^k)_{k\geqslant 1}$ that are part of the infinite step Raiffa negotiation curve that is nevertheless linear - in fact, a line segment joining d to the extreme point of the efficient boundary.

5 A finite horizon bargaining model for the one-shot game

The payoff environment[5] for bargaining is a strategic game G that satisfies Assumption 4. Take the finite horizon bargaining model described by Myerson (1991). This model implements Raiffa's Bargaining Solution in the unique subgame perfect equilibrium in the limit as the model becomes frictionless. Let us modify that bargaining game by adding discounting frictions over and above the deadline frictions that the model already has; and by taking $\mathbf{0}$ as the deadlock point. This bargaining model, denoted by $\mathscr{E}_M(G, \delta, k)$, is depicted in Figure 5.

Let $\mathbf{W}^{(t)}(\delta) = (W_1^{(t)}(\delta), W_2^{(t)}(\delta))$ denote the continuation values of the unique subgame perfect equilibrium of $\mathscr{E}_M(G, \delta, k)$ starting at round t from the end. By the Backward Induction Algorithm, we have

$$\mathbf{W}^{(1)}(\delta) = \mathbf{0} \tag{1}$$

$$\mathbf{W}^{(t+1)}(\delta) = \frac{1}{2}\Big[\Big(h_1\big(\delta W_2^{(t)}(\delta)\big), \delta W_2^{(t)}(\delta)\Big) + \Big(\delta W_1^{(t)}(\delta), h_2\big(\delta W_1^{(t)}(\delta)\big)\Big)\Big] \tag{2}$$

Lemma 7. *Suppose G satisfies Assumption 4. For every $t \in \mathbb{N}$, $\mathbf{W}^{(t)}(\delta)$ is a bounded continuous function of δ.*

Proof. Use mathematical induction. For the base step, equation (1) implies that $\mathbf{W}^{(1)}(\delta)$, as a constant function, is a bounded continuous function of δ. For the inductive step, suppose for every round $1, \ldots, t$ from the end, the claim is true. Then using the fact that h_1 is a bounded continuous map together with the inductive hypothesis for round t and the preservation of continuity by the product and the composition operations in equation (2) implies the claim is true for round $t + 1$ from the end as well. Q.E.D.

Lemma 7, has an immediate consequence:

Corollary 1. *Suppose G satisfies Assumption 4. Then for every $t \in \mathbb{N}$, $\lim_{\delta \to 1} \mathbf{W}^{(t)}(\delta)$ exists and is equal to $\mathbf{W}^{(t)}(1)$.*

With discounting and deadline frictions parameterized as the pair (δ, k) of the common discount factor and the bargaining horizon, we introduce the notion of

Definition 1. *(Noncooperative Raiffa Solution). Define the (δ-discounted, k-period) noncooperative Raiffa solution with respect to the bargaining model $\mathscr{E}_M(G, \delta, k)$ of the strategic*

[5] In Appendix B, we say more about setting up this payoff environment, including stating Assumption 4, in the context of Rubinstein bargaining model with time preference.

game G with status quo point $\mathbf{0}$ to be the unique SPE value $\mathbf{W}^{k+1}(\delta)$ of $\mathscr{E}_M(G,\delta,k)$.

Definition 1 is justified because as the discounting and then the deadline frictions vanish in the bargaining game $\mathscr{E}_M(G,\delta,k)$, the noncooperative Raiffa solution $\mathbf{W}^{k+1}(\delta)$ converges to Raiffa's Bargaining Solution $\mathbf{R}(\mathcal{F},\mathbf{0})$. Formally, we have

$$\lim_{k\to\infty}\lim_{\delta\to 1}\mathbf{W}^{(k)}(\delta)=\lim_{k\to\infty}\mathbf{W}^{(k)}(1)=:\mathbf{R}(\mathcal{F},\mathbf{0})$$

where the first equality is due to Lemma 7.

Define $\mathscr{W}(\delta) := \lim_{t\to\infty}\mathbf{W}^{(t)}(\delta)$ to be the limit[6] of the unique SPE values of $\mathscr{E}_M(G,\delta,k)$ as the deadline frictions vanish in the model. Then taking limits as $t\to\infty$ in equation (2) implies

$$\mathscr{W}(\delta)=\frac{1}{2}\Big[\Big(h_1\big(\delta\mathscr{W}_2(\delta)\big),\delta\mathscr{W}_2(\delta)\Big)+\Big(\delta\mathscr{W}_1(\delta),h_2\big(\delta\mathscr{W}_1(\delta)\big)\Big)\Big] \tag{3}$$

Let $\mathbf{V}(\delta)$ be the unique SPE value of the Rubinstein bargaining model $\mathscr{E}_{tp}(G,\delta)$ with time preference studied in Appendix B. Then Lemma 8 relates the limit (as the deadline frictions vanish) SPE values of $\mathscr{E}_M(G,\delta,k)$ to the SPE values of $\mathscr{E}_{tp}(G,\delta)$.

Lemma 8. *Suppose G satisfies Assumption 4. If $\mathscr{W}(\delta)$ is well defined, then $\mathscr{W}(\delta)=\mathbf{V}(\delta)$.*

Proof. follows from equation (3) and Proposition $B.2$. Q.E.D.

Lemma 9 records the Lipschitz continuity property of the functions describing the efficient boundary of the strategic game. This simple result will be used in the key technical result of this section - Lemma 10. It is an easy consequence of Proposition L which appears as Proposition 2.2.3 in Cobzaş et al. (2019).

Proposition L. *Let $f:[a,b]\mapsto\mathbb{R}$ be continuous. If $f'_-(x)$ exists and is bounded on $(a,b]$ then f is Lipschitz on $[a,b]$ with Lipschitz constant $L(f)=\sup\{|f'_-(x)| : x\in(a,b]\}$. A similar result holds for $f'_+(x)$ on $[a,b)$.*

Lemma 9. *Suppose G satisfies Assumption 4. Then the functions h_1 and h_2 are bounded and Lipschitz continuous on their respective domains $[0,M_2\partial\mathcal{F}]$ and $[0,M_1\partial\mathcal{F}]$.*

Proof. Boundedness is immediate from definitions. Lipschitz continuity follows from Proposition L. Q.E.D.

[6] Corollary 2 of Lemma 10, which is proved later, implies that $\mathscr{W}(\delta)$ is well defined for all sufficiently high δ.

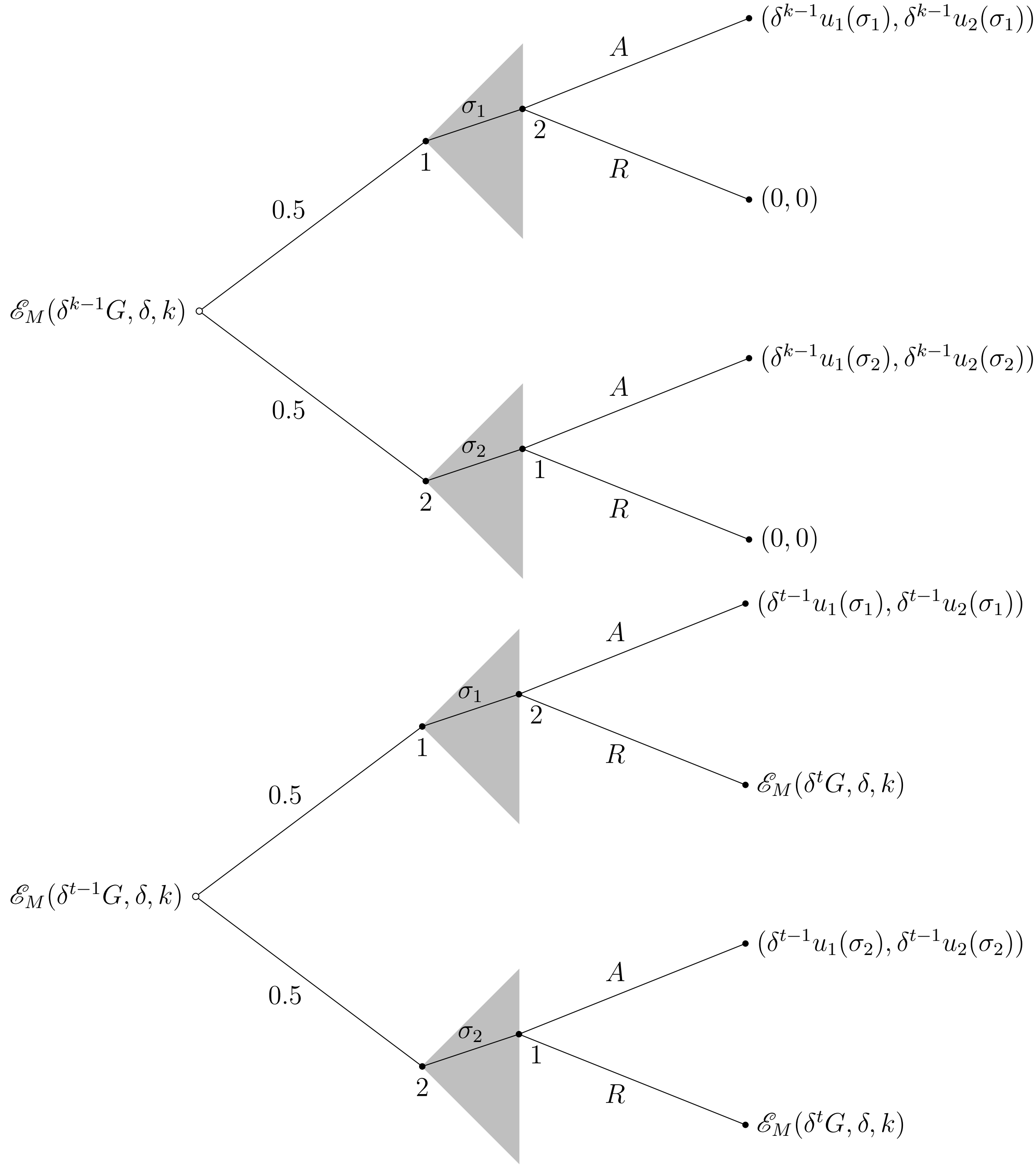

Figure 5: The top panel is the k-th (last) round and the bottom panel is the recursive schema of Myerson (1991) Finite Horizon k-round Bargaining Game $\mathscr{E}_M(G, \delta, k)$ when players have discounted time preferences. For any date $1 \leqslant t < k$, $\delta^t G$ is the strategic game $\langle N, \mathcal{A}, \delta^t \mathbf{u} \rangle$.

Let $L(h_1)$ and $L(h_2)$ be the Lipschitz constants of the mappings h_1 and h_2 respectively. We introduce the assumption that the strategic game G has a linear efficient boundary as

Assumption 1. *(Linear efficient boundary).* $L(h_1)L(h_2) = 1$.

It is well known in the literature that if G satisfies Assumption 1 and Assumption 4. Then Raiffa's Bargaining Solution $\mathbf{R}(\mathcal{F}, \mathbf{d})$ coincides with Nash Bargaining Solution $\mathbf{N}(\mathcal{F}, \mathbf{d})$. We now state and prove the key technical result of this section - as the bargaining model $\mathscr{E}_M(G, \delta, k)$ becomes frictionless, the SPE value functions (as functions of δ) of the model get closer and closer in the sup-metric near $\delta = 1$.

Lemma 10. *Suppose G satisfies Assumption 4. Then there exists $\hat{\delta} \in (0, 1)$ such that $\{\mathbf{W}^{(t)}\}_{t \geqslant 1}$ is a Cauchy sequence of functions from $[\hat{\delta}, 1]$ to $[0, M_1(\partial\mathcal{F})] \times [0, M_2(\partial\mathcal{F})]$ in the metric d_∞ of uniform convergence.*

Proof. We will establish that the lemma holds for both sequences of coordinate functions $\{W_1^{(t)}\}_{t \geqslant 1}$ and $\{W_2^{(t)}\}_{t \geqslant 1}$. For $t, s \in \mathbb{N}$ such that $t > s$, define $\hat{\delta}(t, s) := \operatorname{argmax}_{\delta \in [0,1]} |W_1^{(t+1)}(\delta) - W_1^{(s+1)}(\delta)|$; or equivalently by $d_\infty(W_1^{(t+1)}, W_1^{(s+1)}) = |W_1^{(t+1)}(\hat{\delta}(t, s)) - W_1^{(s+1)}(\hat{\delta}(t, s))|$. For $T \in \mathbb{N}$, define $\hat{\delta}_T := \sup_{t > s \geqslant T} \hat{\delta}(t, s)$.

Suppose it is the case that for every $T \in \mathbb{N}$, we have $\hat{\delta}_T = 1$. This means for every $n \in \mathbb{N}$, there exist $t_n > s_n \geqslant n$ such that $\hat{\delta}(t_n, s_n) \geqslant 1 - \frac{1}{2n}$; or equivalently,

$$\lim_{n \to \infty;\, t_n > s_n \geqslant n} \hat{\delta}(t_n, s_n) = 1 \tag{4}$$

Then

$$\begin{aligned}
\lim_{t,s \to \infty} d_\infty(W_1^{(t+1)}, W_1^{(s+1)}) &= \lim_{t,s \to \infty} |W_1^{(t+1)}(\hat{\delta}(t, s)) - W_1^{(s+1)}(\hat{\delta}(t, s))| \\
&= \lim_{n \to \infty;\, t_n > s_n \geqslant n} |W_1^{(t_n+1)}(\hat{\delta}(t_n, s_n)) - W_1^{(s_n+1)}(\hat{\delta}(t_n, s_n))| \\
&= \lim_{n \to \infty;\, t_n > s_n \geqslant n} |W_1^{(t_n+1)}(1) - W_1^{(s_n+1)}(1)| \qquad (5) \\
&= 0 \qquad (6)
\end{aligned}$$

where (5) follows because of (4); and, (6) follows because $\{W_1^{(t)}(1)\}_t$, as a sequence of real numbers that converges (Myerson (1991)) to $\mathrm{RBS}_1(\mathcal{F}, \mathbf{0})$, must be Cauchy. This gives the conclusion of the lemma in the present case.

Otherwise, there exists a $T \in \mathbb{N}$ such that $\hat{\delta}_T < 1$. For arbitrary t and s such that $t > s$, using the recursive definition (2), write the point-wise difference between the functions $W_1^{(t+1)}$ and $W_1^{(s+1)}$ as

$$
\begin{aligned}
|W_1^{(t+1)}(\delta) - W_1^{(s+1)}(\delta)| &= \frac{1}{2}\big|h_1\big(\delta W_2^{(t)}(\delta)\big) - h_1\big(\delta W_2^{(s)}(\delta)\big)\big| + \frac{\delta}{2}|W_1^{(t)}(\delta) - W_1^{(s)}(\delta)| \\
&\leqslant \frac{\delta L(h_1)}{2}|W_2^{(t)}(\delta) - W_2^{(s)}(\delta)| + \frac{\delta}{2}|W_1^{(t)}(\delta) - W_1^{(s)}(\delta)| \qquad (7) \\
&\leqslant \frac{\delta L(h_1)}{2} d_\infty(W_2^{(t)}, W_2^{(s)}) + \frac{1}{2} d_\infty(W_1^{(t)}, W_1^{(s)}) \qquad (8)
\end{aligned}
$$

where (7) follows from Lemma 9. This gives the estimate

$$
\begin{aligned}
d_\infty(W_1^{(t+1)}, W_1^{(s+1)}) &= |W_1^{(t+1)}(\hat{\delta}(t,s)) - W_1^{(s+1)}(\hat{\delta}(t,s))| \\
&\leqslant \frac{\hat{\delta}(t,s)L(h_1)}{2} d_\infty(W_2^{(t)}, W_2^{(s)}) + \frac{1}{2} d_\infty(W_1^{(t)}, W_1^{(s)}) \qquad \text{(using (8))} \qquad (9) \\
&\leqslant \frac{\hat{\delta}_T L(h_1)}{2} d_\infty(W_2^{(t)}, W_2^{(s)}) + \frac{1}{2} d_\infty(W_1^{(t)}, W_1^{(s)}) \qquad (10)
\end{aligned}
$$

Arguing on parallel lines, we obtain the estimate

$$
d_\infty(W_2^{(t+1)}, W_2^{(s+1)}) \leqslant \frac{1}{2} d_\infty(W_1^{(t)}, W_1^{(s)}) + \frac{\hat{\delta}_T L(h_2)}{2} d_\infty(W_2^{(t)}, W_2^{(s)}) \qquad (11)
$$

From (10) and (11), we obtain

$$
\lim_{t,s\to\infty} d_\infty(W_1^{(t)}, W_1^{(s)}) \leqslant \hat{\delta}_T L(h_1) \lim_{t,s\to\infty} d_\infty(W_2^{(t)}, W_2^{(s)}) \qquad (12)
$$

$$
\lim_{t,s\to\infty} d_\infty(W_2^{(t)}, W_2^{(s)}) \leqslant \hat{\delta}_T L(h_2) \lim_{t,s\to\infty} d_\infty(W_1^{(t)}, W_1^{(s)}) \qquad (13)
$$

(12) and (13) give the conclusion

$$
(1 - \hat{\delta}_T^2 L(h_1)L(h_2)) \lim_{t,s\to\infty} d_\infty(W_1^{(t)}, W_1^{(s)}) \leqslant 0 \qquad (14)
$$

$$
(1 - \hat{\delta}_T^2 L(h_1)L(h_2)) \lim_{t,s\to\infty} d_\infty(W_2^{(t)}, W_2^{(s)}) \leqslant 0 \qquad (15)
$$

Consider two cases. For the first case, suppose $\mathbf{N}(\mathcal{F}, \mathbf{0})$ is not an extreme point of $\mathcal{F}$. Under the hypotheses of the present lemma, by Lemma 8 and Proposition $B.4$, there exists $\hat{\delta} \in (0,1)$ such that for every $\delta \geqslant \hat{\delta}$, there exists $T_\delta \in \mathbb{N}$ such that for every $t \geqslant T_\delta$, the points $(h_1(\delta W_2^{(t)}(\delta)), \delta W_2^{(t)}(\delta))$ and $(\delta W_1^{(t)}(\delta), h_2(\delta W_1^{(t)}(\delta)))$ lie on the same linear edge of the efficient boundary $\partial\mathcal{F}$ (See the left panel in Figure 6). This implies Assumption 1 is satisfied for the mappings h_1 when restricted to domain $[\delta W_2^{(t)}(\delta), h_2(\delta W_1^{(t)}(\delta))]$ and h_2 when

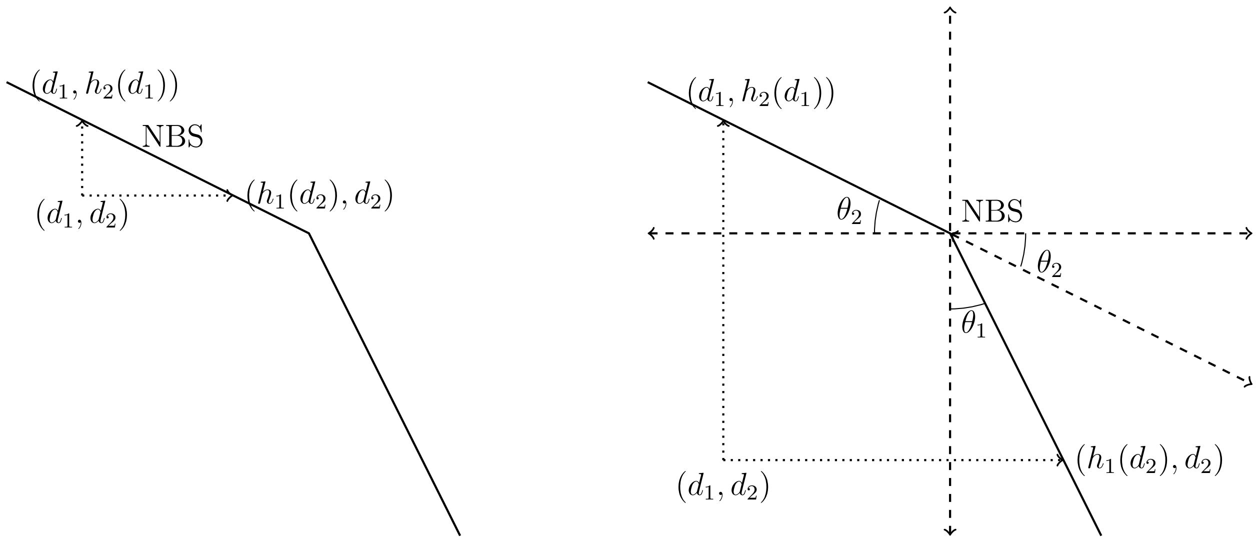

Figure 6: Left Panel: NBS is at a point on the edge of the game polytope. For the given disagreement point d, the mappings h_1 and h_2 correspond to the same edge of the polytope. Right Panel: NBS is an extreme point of the game polytope. For the given disagreement point d, the mappings h_1 and h_2 correspond to the adjacent edges of the polytope that meet at the NBS.

restricted to domain $[\delta W_1^{(t)}(\delta), h_1(\delta W_2^{(t)}(\delta))]$ with $\delta = \hat{\delta}$ and $t = T_{\hat{\delta}}$. Using this in (14) and (15) gives the conclusion of Lemma 10 under the first case.

For the second case, suppose $\mathbf{N}(\mathcal{F}, \mathbf{0})$ is an extreme point of $\mathcal{F}$. In this case, by Lemma 8 and Proposition $B.4$ again, there exists $\hat{\delta} \in (0, 1)$ such that for every $\delta \geqslant \hat{\delta}$, there exists $T_\delta \in \mathbb{N}$ such that for every $t \geqslant T_\delta$, the points $(h_1(\delta W_2^{(t)}(\delta)), \delta W_2^{(t)}(\delta))$ and $(\delta W_1^{(t)}(\delta), h_2(\delta W_1^{(t)}(\delta)))$ now lie on those adjacent linear edges of the efficient boundary $\partial\mathcal{F}$, which meet at the given extreme point. This again means that the relevant mappings are given by h_1 when restricted to domain $[\delta W_2^{(t)}(\delta), \mathrm{NBS}_2(\mathcal{F}, \mathbf{0})]$ and h_2 when restricted to domain $[\delta W_1^{(t)}(\delta), \mathrm{NBS}_1(\mathcal{F}, \mathbf{0})]$ with $\delta = \hat{\delta}$ and $t = T_{\hat{\delta}}$. Guided by the right panel in Figure 6, the product of the involved Lipschitz constants of these mappings with restricted domains is

$$\tan\theta_2 \tan\theta_1 < \tan\theta_2 \tan(90^\circ - \theta_2) = 1 \quad \text{for any } \theta_2 \in (0, 90^\circ) \tag{16}$$

where the inequality in (16) is because $\theta_1 < 90^\circ - \theta_2$ owing to the shape of the efficient boundary $\partial\mathcal{F}$ near an extreme point which must look like as shown in the right panel in Figure 6; and the equality in (16) is because in the trigonometric identity

$$\tan(\theta_2 + (90^\circ - \theta_2)) = \frac{\tan\theta_2 + \tan(90^\circ - \theta_2)}{1 - \tan\theta_2 \tan(90^\circ - \theta_2)}$$

the left hand side is ∞ while the numerator of the fraction on the right hand side is finite and positive. (16) says that for mappings h_1 and h_2 with the aforementioned restricted domains, we have the product of their Lipschitz constants $L(h_1)L(h_2) < 1$. Using this in (14) and (15) gives the conclusion of Lemma 10 under the second case as well. Q.E.D.

Since uniform convergence implies pointwise convergence, Lemma 10 has the following easy consequence:

Corollary 2. *Suppose G satisfies Assumption 4. Then there exists $\hat{\delta} \in (0,1)$ such that $\{\mathbf{W}^{(t)}\}_{t\geqslant 1}$ is a sequence of functions that converges pointwise on $[\hat{\delta}, 1]$.*

Note that Corollary 1 implies that for every t, $\lim_{\delta\to 1} \mathbf{W}^{(t)}(\delta)$ exists and Corollary 2 implies that for all δ sufficiently high, $\lim_{t\to\infty} \mathbf{W}^{(t)}(\delta)$ exists. We now have the main result in this section.

Theorem 1. *Suppose G satisfies Assumption 4. Then as the bargaining frictions disappear, the noncooperative Raiffa solution $\mathbf{W}^{(t)}(\delta)$ with respect to $\mathscr{E}_M(G, \delta, k)$ converges to the Nash Bargaining Solution $\boldsymbol{N}(\mathcal{F}, \mathbf{0})$.*

Proof. We have

$$\begin{aligned}
\lim_{t\to\infty}\lim_{\delta\to 1} \mathbf{W}^{(t)}(\delta) &= \lim_{\delta\to 1}\lim_{t\to\infty} \mathbf{W}^{(t)}(\delta) && (\mathbf{W}^{(t)} \text{ converges uniformly near 1 by Lemma 10}) \\
&= \lim_{\delta\to 1} \mathscr{W}(\delta) && (\text{By definition of } \mathscr{W}(\delta)) \\
&= \lim_{\delta\to 1} \mathbf{V}(\delta) && (\text{By Lemma 8}) \\
&= \mathbf{N}(\mathcal{F}, \mathbf{0}) && (\text{By Proposition } B.4)
\end{aligned}$$

Q.E.D.

6 A bargaining model for the infinitely repeated game with a status quo

In this section, which is an interlude to the main theme, we study a random-offer Rubinstein style model but for an infinitely repeated game with an initial status quo contract. We have two objectives. One, to record a convergence-to-Nash result, which will be called in the proof of our main result, Theorem 2; and two, to examine the link between Raiffa's mediation interpretation of his solution with the strategic aspects of the bargaining situation. Since this latter objective is somewhat tangential to our main theme, we relegate it to Appendix C.

Let $G = \langle\{1,2\},(\mathcal{A}_i,u_i)_{i=1,2}\rangle$ be a strategic game. Let $\sigma^0 \in \Delta(\mathcal{A})$ be exogenously given and to be interpreted as the initial contractual state. Let $\mathbf{u} = (u_1, u_2)$ be the profile of representing vNM utility functions of the players. We refer to the extensive form bargaining model with flow payoffs that we are about to describe as $\mathscr{E}(G^\infty, \sigma^0, \delta)$.

Dates. The model has a discrete set of dates $t = 1, 2, \ldots$. The time between date t and date $t+1$ is referred to as period t.

States. Specify the state space for the model as the set of feasible contracts $\Delta(\mathcal{A})$, and $\sigma^0 \in \Delta(\mathcal{A})$ as the initial state of the model. Specify any efficient contract - that is, any contract $\sigma \in \Delta(\mathcal{A})$ such that $\mathbf{u}(\sigma) \in \partial\mathcal{F}$ - as a stopped state for the model. A state that is not a stopped state of the model is an active state.

Actions. Actions are available to players only in active states. At any date at which the model is in an active state, a player is randomly chosen with probability $1/2$ to be the proposer. The other player becomes the responder. The date-invariant set of feasible actions for a player in the role of a proposer is the set $\Delta(\mathcal{A})$ of feasible contracts in G. The date-invariant set of feasible actions for a player in the role of a responder is $\{A, R\}$ denoting 'accept the proposed contract' and 'reject the proposed contract' respectively.

State Transitions. Consider the model in an active state σ at some date t. If a different state σ' is proposed and accepted, then the state of the model changes to σ' at the end of period t. Otherwise, the model stays in state σ. Stopped states are absorbing.

Payoffs. Every period, players realize their payoffs at the end when actions have been taken and state transitions (if any) have been effected. The payoff vector $\mathbf{v}_t$ realized by players in period t when the model is in an active state σ_t, the offer made by the proposer is σ'_t, and the response of the other player is r_t is given by

$$\mathbf{v}_t(\sigma_t, \sigma'_t, r_t) = \begin{cases} \mathbf{u}(\sigma_t) \text{ if } r_t = R \\ \mathbf{u}(\sigma'_t) \text{ if } r_t = A \end{cases} \qquad \text{for } \sigma_t \text{ active}\ ; \quad \mathbf{v}_t(\sigma) = \mathbf{u}(\sigma) \quad \text{for } \sigma \text{ stopped}$$

The associated bargaining problem for the strategic bargaining model $\mathscr{E}(G^\infty, \sigma^0, \delta)$ is given by $(\mathcal{F}, \mathbf{u}(\sigma^0))$. We maintain the following

Assumption 2. $\mathbf{u}(\sigma^0)) \geqslant \mathbf{m}(\partial\mathcal{F})$

Stationary Markov Strategies A deterministic stationary Markov bargaining strategy b_i for player i in the model is a pair (f_i, r_i) of offer and response strategies that in any period t

make reference only to the current state σ_t and in doing so, do not depend on the date t. Player i's offer strategy is a state contingent rule for making offers and formally specified as a function $f_i : \Delta(\mathcal{A}) \mapsto \Delta(\mathcal{A})$. $f_i(\sigma)$ is the offer made by player i as a proposer when the state is σ irrespective of what date it is. Player i's response strategy is a state contingent rule for responding to offers and formally specified as a function $r_i : \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \mapsto \{A, R\}$. $r_i(\sigma, \sigma')$ is the binary response of player i as a responder when the state is σ and the offer made to her is σ' irrespective of what date it is.

Solution Concept. A profile of stationary Markov bargaining strategies $\mathbf{b} = (b_1, b_2)$ is a subgame perfect equilibrium (SMPE) of the bargaining model if they constitute a Nash equilibrium in every state of the model.

Let $\mathbf{V}(\sigma, \mathbf{b}; \delta) = (V_1(\sigma, \mathbf{b}; \delta), V_2(\sigma, \mathbf{b}; \delta)) \in \mathbb{R}^2$ be the value vector in state σ of the bargaining strategy profile $\mathbf{b}$ when the common discount factor is δ. A value function for the model is a function $\mathbf{W} : \Delta(\mathcal{A}) \mapsto \mathcal{F}$ that maps model states to a pair of feasible values for the players in the underlying game G. Let $\mathscr{V}$ be the space of value functions for the model. We now introduce two distinguished value functions in $\mathscr{V}$ and then an operator on $\mathscr{V}$.

Let $\mathbf{N}(\mathcal{F}, \bullet) : \Delta(\mathcal{A}) \mapsto \mathcal{F}$ be the Nash bargaining solution function that for any state σ, gives the Nash bargaining solution of the bargaining problem $(\mathcal{F}, \mathbf{u}(\sigma))$. Formally, it is defined by

$$\mathbf{N}(\mathcal{F}, \sigma) = \underset{\mathbf{u} \in \mathcal{F}}{\operatorname{argmax}} (u_1 - u_1(\sigma))(u_2 - u_2(\sigma)) \tag{17}$$

Define the *Raiffa operator* $\mathscr{R} : \mathscr{V} \mapsto \mathscr{V}$ on the space of value functions of the model so that for any value function $\mathbf{V} \in \mathscr{V}$ and at any state σ, it takes the value

$$\mathscr{R}\mathbf{V}(\sigma) = \frac{1}{2}\Big[\big(h_1(V_2(\sigma)), V_2(\sigma)\big) + \big(V_1(\sigma), h_2(V_1(\sigma))\big)\Big] \tag{18}$$

Note that Raiffa's bargaining solution from a status quo contract σ is the iterative limit under the Raiffa operator of the vNM payoffs in state σ. In other words, $\mathbf{RBS}(\mathcal{F}, \sigma) := \lim_{n\to\infty} \mathscr{R}^n \mathbf{u}(\sigma)$.

Proposition 1 describes the stationary Markov bargaining equilibria in the bargaining model with flow payoffs in terms of strategy optimality and value optimality in a manner that parallels Proposition $A.1$ for the risk-of-breakdown model. It follows from standard dynamic programming arguments applied to the given model. In the statement for value optimality, $\mathscr{B} = (\mathscr{B}_1, \mathscr{B}_2)$ is the Bellman operator on $\mathscr{V}$.

Proposition 1. *(Stationary Markov Perfect Equilibria). A profile of stationary Markov bargaining strategies* $\mathbf{b} = (b_1, b_2)$ *where* $b_i = (f_i, r_i)$ *for* $i = 1, 2$ *constitutes a subgame-perfect* *equilibrium of the bargaining model $\mathscr{E}(G^\infty, \sigma^0, \delta)$ if and only if*

(i) (Strategy Optimality). Given the associated pair of value functions $V_1(\bullet, \mathbf{b}; \delta) : \Delta(\mathcal{A}) \mapsto \mathbb{R}$ and $V_2(\bullet, \mathbf{b}; \delta) : \Delta(\mathcal{A}) \mapsto \mathbb{R}$, we have
(a) (Response Optimality). For every player $i = 1, 2$, for every state σ, and for every offer σ'

$$r_i(\sigma, \sigma') = A \text{ if and only if } (1-\delta)u_i(\sigma') + \delta V_i(\sigma', \mathbf{b}; \delta) \geqslant (1-\delta)u_i(\sigma) + \delta V_i(\sigma, \mathbf{b}; \delta)$$

(b) (Offer Optimality). For every state σ, player 1's offer $f_1(\sigma)$ and player 2's offer $f_2(\sigma)$ are such that

$$(1-\delta)u_1(f_1(\sigma)) + \delta V_1(f_1(\sigma), \mathbf{b}; \delta) = h_1\big((1-\delta)u_2(\sigma) + \delta V_2(\sigma, \mathbf{b}; \delta)\big) \tag{19}$$
$$(1-\delta)u_2(f_2(\sigma)) + \delta V_2(f_2(\sigma), \mathbf{b}; \delta) = h_2\big((1-\delta)u_1(\sigma) + \delta V_1(\sigma, \mathbf{b}; \delta)\big) \tag{20}$$

(ii) (Value Optimality). The associated pair of value functions $V_1(\bullet, \mathbf{b}; \delta)$ and $V_2(\bullet, \mathbf{b}; \delta)$ satisfy the Bellman equations of optimality $\mathscr{B}(\mathbf{V}) = \mathbf{V}$. In other words, for every state σ,

$$V_1(\sigma; \delta) = (\mathscr{B}_1\mathbf{V})(\sigma) := \frac{h_1\big((1-\delta)u_2(\sigma) + \delta V_2(\sigma; \delta)\big) + (1-\delta)u_1(\sigma) + \delta V_1(\sigma; \delta)}{2} \tag{21}$$
$$V_2(\sigma; \delta) = (\mathscr{B}_2\mathbf{V})(\sigma) := \frac{(1-\delta)u_2(\sigma) + \delta V_2(\sigma; \delta) + h_2\big((1-\delta)u_1(\sigma) + \delta V_1(\sigma; \delta)\big)}{2} \tag{22}$$

Proposition 2. *(Uniqueness and Convergence of SMPE Value Functions). Suppose G satisfies Assumption 2. For any $\delta \in [0, 1)$, the Bellman equations of optimality $\mathscr{B}(\mathbf{V}) = \mathbf{V}$, specified in Proposition 1, have a unique solution $\mathbf{V}(\bullet; \delta) : \Delta(\mathcal{A}) \mapsto \mathcal{F}$. Moreover, $\lim_{\delta\to 1} \mathbf{V}(\bullet; \delta) = \mathbf{N}(\mathcal{F}, \bullet)$ and $\mathbf{V}(\bullet; 0) = \mathscr{R}\mathbf{u}(\bullet)$.*

Proof. For every fixed δ and every fixed state σ, the Bellman equations (21) and (22) are mathematically equivalent to the Bellman equations in the risk-of-breakdown model. Proposition $A.2$ in that model, therefore, implies that (21) and (22) have a unique solution $\mathbf{W}(\sigma; \delta) = (W_1(\sigma; \delta), W_2(\sigma; \delta)) \in \mathcal{F}$. For every fixed δ, construct the function $\mathbf{V}(\bullet; \delta)$ by setting $\mathbf{V}(\sigma; \delta) = \mathbf{W}(\sigma; \delta)$ for every state σ. The convergence conclusion follows because Proposition $A.4$ in the risk-of-breakdown model implies $\mathbf{V}(\bullet; \delta)$ converges pointwise in σ to $\mathbf{N}(\mathcal{F}, \bullet)$ as $\delta \to 1$. The last assertion follows from the Bellman equations at $\delta = 0$ and the definition of Raiffa operator in (18). Q.E.D.

Proposition 2 says that in the bargaining model with flow payoffs, the optimal value function is the Raiffa value function when players are impatient. However, when they are arbitrarily patient, the optimal value function approaches the Nash bargaining solution function.

7 A bargaining model for the finitely repeated game with a status quo

Let $G = \langle \{1,2\}, (\mathcal{A}_i, u_i)_{i=1,2} \rangle$ be the underlying strategic game. Let σ^0 be an exogenous initial contract. Let $\mathbf{u} = (u_1, u_2)$ be the profile of representing vNM utility functions of the players. We refer to the bargaining model with per-period payoffs that we are about to describe as $\mathscr{E}(G^k, \sigma^0, \delta)$.

Dates. The model has a finite set of dates $t = k+1, \ldots, 1$ with $k \in \mathbb{N}$. The labeling of dates is backwards from the terminal date. The time between date $t+1$ and date t is referred to as period t. So the model has k periods.

States. Specify the state space for the model as the set of feasible contracts $\Delta(\mathcal{A})$, and σ^0 as the initial state of the model. Specify any efficient contract - that is, any contract $\sigma \in \Delta(\mathcal{A})$ such that $\mathbf{u}(\sigma) \in \partial\mathcal{F}$ - as a stopped state for the model. A state that is not a stopped state of the model is an active state.

Actions. Actions are available to players only in active states and at non-terminal dates $t \neq 1$. At any non-terminal date where the model is in an active state, a player is randomly chosen with probability $1/2$ to be the proposer. The other player becomes the responder. The date-invariant set of feasible actions for a player in the role of a proposer is the set $\Delta(\mathcal{A})$ of feasible contracts in G. The date-invariant set of feasible actions for a player in the role of a responder is $\{A, R\}$ denoting 'accept the proposed contract' and 'reject the proposed contract' respectively.

State Transitions. Consider the model in an active state σ at some date t. If a different state σ' is proposed and accepted, then the state of the model changes to σ' at the end of period t. Otherwise, the model stays in state σ. Stopped states are absorbing.

Payoffs. Every period, players realize their payoffs at the end when actions have been taken and state transitions (if any) have been effected. The payoff vector $\mathbf{v}_t$ realized by players in period t when the model is in an active state σ_t, the offer made by the proposer is σ'_t, and the response of the other player is r_t is given by

$$\mathbf{v}_t(\sigma_t, \sigma'_t, r_t) = \begin{cases} \mathbf{u}(\sigma_t) \text{ if } r_t = R \\ \mathbf{u}(\sigma'_t) \text{ if } r_t = A \end{cases} \qquad \text{for } \sigma_t \text{ active ;} \quad \mathbf{v}_t(\sigma) = \mathbf{u}(\sigma) \quad \text{for } \sigma \text{ stopped}$$

Construct the associated Nash bargaining problem $(\mathcal{F}, \mathbf{u}(\sigma^0))$ so that it satisfies Assumption 2.

Markov Strategies. A Markov bargaining strategy b_i for player i in the model is a sequence $(f_i^t, r_i^t)_{t=1}^k$ of offer and response strategies that in any period t make reference only to the current state σ_t. Player i's offer strategy at date t is a state contingent rule for making offers and formally specified as a function $f_i^t : \Delta(\mathcal{A}) \mapsto \Delta(\mathcal{A})$. $f_i^t(\sigma)$ is the offer made by player i as a proposer when the state is σ at date t. Player i's response strategy at date t is a state contingent rule for responding to offers and formally specified as a function $r_i^t : \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \mapsto \{A, R\}$. $r_i^t(\sigma, \sigma')$ is the binary response of player i as a responder when the state is σ and the offer made to her is σ' at date t.

Solution Concept. A profile of Markov bargaining strategies $\mathbf{b} = (b_1, b_2)$ is a subgame perfect equilibrium (MPE) of the bargaining model if they constitute a Nash equilibrium at every date in every state of the model.

Let $\mathbf{V}^t(\sigma, \mathbf{b}; \delta) = (V_1^t(\sigma, \mathbf{b}; \delta), V_2^t(\sigma, \mathbf{b}; \delta)) \in \mathbb{R}^2$ be the value vector in state σ at date t of the bargaining strategy profile $\mathbf{b}$ when the common discount factor is δ. A date t value function for the model is a function $\mathbf{W}^t : \Delta(\mathcal{A}) \mapsto \mathcal{F}$ that maps model states to a pair of feasible values for the players in the underlying game G.

Proposition 3 describes the Markov bargaining equilibria in the finitely repeated game model in terms of strategy optimality and value optimality in a manner that parallels Proposition $A.1$ for the risk-of-breakdown model. It follows from standard dynamic programming arguments applied to the given model.

Proposition 3. *(Markov Perfect Equilibria). A profile of Markov bargaining strategies* $\mathbf{b} = (b_1, b_2)$ *where* $b_i = (f_i^t, r_i^t)_{t=2}^{k+1}$ *for* $i = 1, 2$ *constitutes a subgame-perfect equilibrium of the bargaining model* $\mathscr{E}(G^k, \sigma^0, \delta)$ *if and only if*

(i) (Strategy Optimality). Given the associated sequence of value functions $V_1^t(\bullet, \mathbf{b}; \delta) : \Delta(\mathcal{A}) \mapsto \mathbb{R}$ *and* $V_2^t(\bullet, \mathbf{b}; \delta) : \Delta(\mathcal{A}) \mapsto \mathbb{R}$ *for every date* $t = 1, \ldots, k+1$*, we have*
(a) (Response Optimality). For every player $i = 1, 2$*, for every* $t \neq k+1$*, for every state* σ*,*

and for every offer σ'

$$r_i^{t+1}(\sigma,\sigma') = A \text{ if and only if } (1-\delta)u_i(\sigma') + \delta V_i^t(\sigma',\mathbf{b};\delta) \geqslant (1-\delta)u_i(\sigma) + \delta V_i^t(\sigma,\mathbf{b};\delta)$$

(b) (Offer Optimality). For every $t \neq k+1$, *and for every state* σ, *player 1's offer* $f_1^{t+1}(\sigma)$ *and player 2's offer* $f_2^{t+1}(\sigma)$ *are such that*

$$(1-\delta)u_1(f_1^{t+1}(\sigma)) + \delta V_1^t(f_1^{t+1}(\sigma),\mathbf{b};\delta) = h_1\big((1-\delta)u_2(\sigma) + \delta V_2^t(\sigma,\mathbf{b};\delta)\big) \tag{23}$$
$$(1-\delta)u_2(f_2^{t+1}(\sigma)) + \delta V_2^t(f_2^{t+1}(\sigma),\mathbf{b};\delta) = h_2\big((1-\delta)u_1(\sigma) + \delta V_1^t(\sigma,\mathbf{b};\delta)\big) \tag{24}$$

(ii) (Value Optimality). The sequence of associated value functions $(V_1^t(\bullet,\mathbf{b};\delta), V_2^t(\bullet,\mathbf{b};\delta))_{t=1}^{k+1}$ *satisfy the Bellman equations of optimality at every date and in every state. In other words, for every* $t \neq k+1$, *and for every state* σ,

$$V_1^{t+1}(\sigma;\delta) = \frac{h_1\big((1-\delta)u_2(\sigma) + \delta V_2^t(\sigma;\delta)\big) + (1-\delta)u_1(\sigma) + \delta V_1^t(\sigma;\delta)}{2} \tag{25}$$
$$V_2^{t+1}(\sigma;\delta) = \frac{(1-\delta)u_2(\sigma) + \delta V_2^t(\sigma;\delta) + h_2\big((1-\delta)u_1(\sigma) + \delta V_1^t(\sigma;\delta)\big)}{2} \tag{26}$$
$$\mathbf{V}^1(\sigma;\delta) = \mathbf{u}(\sigma) \qquad (\textit{Boundary Condition}) \tag{27}$$

Proposition 4. *(Uniqueness of Subgame Perfect Equilibrium Values). All Markov perfect equilibria (MPE) of the bargaining model* $\mathscr{E}(G^k,\sigma^0,\delta)$ *share the same date sequence* $(\mathbf{V}^t(\bullet;\delta))_{t=1}^{k+1}$ *of continuation value functions. Moreover,* $(\mathbf{V}^t(\bullet;\delta))_{t=1}^{k+1}$ *is also the unique date sequence of continuation value functions shared by all subgame perfect equilibria (SPE) of the model.*

Proof. The uniqueness of MPE value function follows by using mathematical induction on the date variable (backwards from the terminal date) in the value optimality conditions of Proposition 3.

For the second statement of the proposition, first note that the SPE of the model are characterized by a proposition analogous to Proposition 3 which characterizes the MPE, but with the modification that the strategies and their associated value functions may potentially be history dependent. Nevertheless, the SPE value functions must satisfy the potentially history dependent versions[7] of the Bellman equations of optimality at every date and for every history, including the boundary conditions at the terminal date for every history.

[7] For reasons of economizing space, we avoid writing these formally because we would need to develop notation for histories and history dependent strategies and value functions but without much added benefit in terms of results.

Now use mathematical induction on the date variable (backwards from the terminal date). For the base step, the value functions of all SPEs must satisfy the boundary condition at the terminal date for every history. Consequently, their continuation value for $t = 1$ is the same - $\mathbf{u}(\sigma)$ - which depends on the history only through the current state σ. For the inductive step, suppose all SPEs share the same date sequence of continuation value functions $(\mathbf{V}^n(\bullet;\delta))_{n=1}^t$, all of which depend on history only through the current state σ. Then the date $t+1$ continuation value functions of all SPEs must satisfy the date $t+1$ Bellman equations - the potentially history dependent versions of (25) and (26). Now on any history leading upto date $t+1$ and ending in state σ, the expressions on the right side of these Bellman equations evaluate uniquely and by induction hypothesis, depend on the history only through the current state σ. This implies all SPEs share the same date $t+1$ continuation value function which depends on the history only through the current state. The SPE strategies then inherit the Markovian nature of the SPE value functions. Q.E.D.

With discounting and deadline frictions parameterized as the pair (δ, k), we define

Definition 2. *(Noncooperative Raiffa Solution). Define the (δ-discounted, k-period) noncooperative Raiffa solution with respect to the bargaining model $\mathscr{E}(G^k, \sigma^0, \delta)$, of the strategic game G with initial contractual state σ^0, to be the SPE equilibrium value $\mathbf{V}^{k+1}(\sigma^0;\delta)$ of $\mathscr{E}(G^k, \sigma^0, \delta)$.*

Example 1. Suppose we have a strategic game G for which $\mathcal{F} = \{\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2 : u_1 + 2u_2 \leqslant 12, 2u_1 + u_2 \leqslant 12\}$; and the initial contractual state σ^0 is such that $\mathbf{u}(\sigma^0) = (2,1)$. For this bargaining problem $(\mathcal{F}, \mathbf{u}(\sigma^0))$, the Nash bargaining solution is $(4,4)$ and the Kalai Smorodinsky solution is $(4.2275, 3.545)$. We apply Proposition 3 to compute the 3-period noncooperative Raiffa solution of the bargaining model $\mathscr{E}(G^3, \sigma^0, \delta)$ with initial contractual state σ^0.

Boundary Condition. For this example, we have the boundary condition (27), namely $\mathbf{V}^1(\sigma^0;\delta) = \mathbf{u}(\sigma^0) = (2,1)$.

First Period. In the first period (t=2), suppose the prevailing state is σ. Player 1, when chosen as a proposer, chooses an offer $f_1^2(\sigma)$ that satisfies the following two conditions: (a) makes player 2 (the responder) indifferent between accepting and rejecting the offer; and (b) is optimal for player 1 subject to acceptance. Using response optimality and offer optimality in Proposition 3, these conditions are formally expressed as

$$(1-\delta)u_2(f_1^2(\sigma)) + \delta V_2^1(f_1^2(\sigma);\delta) = (1-\delta)u_2(\sigma) + \delta V_2^1(\sigma;\delta) \tag{28}$$

$$(1-\delta)u_1(f_1^2(\sigma)) + \delta V_1^1(f_1^2(\sigma);\delta) = h_1((1-\delta)u_2(\sigma) + \delta V_2^1(\sigma;\delta)) \tag{29}$$

Using the boundary condition (27), conditions (28) and (29) simplify to $\mathbf{u}(f_1^2(\sigma)) = (h_1(u_2(\sigma)), u_2(\sigma))$. Similarly, player 2, when chosen as a proposer, chooses an offer $f_2^2(\sigma)$ that satisfies $\mathbf{u}(f_2^2(\sigma)) = (u_1(\sigma), h_2(u_1(\sigma)))$. Using value optimality in Proposition 3, the continuation value vector of the model from date $t = 2$ onwards in state σ is given by

$$\begin{aligned}\mathbf{V}^2(\sigma;\delta) &= \Big(\frac{u_1(f_1^2(\sigma)) + u_1(f_2^2(\sigma))}{2}, \frac{u_2(f_1^2(\sigma)) + u_2(f_2^2(\sigma^0))}{2}\Big) \\ &= \Big(\frac{h_1(u_2(\sigma)) + u_1(\sigma)}{2}, \frac{u_2(\sigma) + h_2(u_1(\sigma))}{2}\Big) \end{aligned} \tag{30}$$

Computing this in the context of the present example gives

$$\mathbf{V}^2(\sigma^0;\delta) = \Big(\frac{\frac{12-u_2(\sigma^0)}{2} + u_1(\sigma^0)}{2}, \frac{u_2(\sigma^0) + \frac{12-u_1(\sigma^0)}{2}}{2}\Big) = (3.75, 3) \tag{31}$$

Second Period. Repeat the argument for the second period. In period 2 (t=3), suppose the prevailing state is σ. Player 1, when chosen as a proposer, chooses an offer $f_1^3(\sigma)$ that satisfies

$$(1-\delta)u_2(f_1^3(\sigma)) + \delta V_2^2(f_1^3(\sigma);\delta) = (1-\delta)u_2(\sigma) + \delta V_2^2(\sigma;\delta) \tag{32}$$

$$(1-\delta)u_1(f_1^3(\sigma)) + \delta V_1^2(f_1^3(\sigma);\delta) = h_1((1-\delta)u_2(\sigma) + \delta V_2^2(\sigma;\delta)) \tag{33}$$

Similarly, Player 2, when chosen as a proposer, chooses an offer $f_2^3(\sigma)$ that satisfies

$$(1-\delta)u_1(f_2^3(\sigma)) + \delta V_2^1(f_2^3(\sigma);\delta) = (1-\delta)u_1(\sigma) + \delta V_1^2(\sigma;\delta) \tag{34}$$

$$(1-\delta)u_2(f_2^3(\sigma)) + \delta V_2^2(f_2^3(\sigma);\delta) = h_2((1-\delta)u_2(\sigma) + \delta V_1^2(\sigma;\delta)) \tag{35}$$

Note that the *the prevailing contract σ^0 at the current date $t = 3$ determines the future contract (either $f_1^3(\sigma)$ or $f_2^3(\sigma)$) at date $t = 2$* by determining the continuation value of rejection in the current period for both players. This continuation value of rejection is a weighted average of payoffs accruing to players from two channels - the current period vNM payoff from the prevailing contract σ, and the future (date 2 onwards) continuation value $\mathbf{V}^2(\sigma;\delta)$ of the prevailing contract σ. The continuation value vector of the model from date $t = 3$ onwards in state σ is then given by

$$\mathbf{V}^3(\sigma;\delta) = (V_1^3(\sigma;\delta), V_2^3(\sigma;\delta)); \text{ where} \tag{36}$$

$$V_1^3(\sigma;\delta) = \frac{(1-\delta)u_1(f_1^3(\sigma)) + \delta V_1^2(f_1^3(\sigma);\delta) + (1-\delta)u_1(f_2^3(\sigma)) + \delta V_1^2(f_2^3(\sigma);\delta)}{2}$$

$$= \frac{h_1((1-\delta)u_2(\sigma)+\delta V_2^2(\sigma;\delta)) + (1-\delta)u_1(\sigma)+\delta V_1^2(\sigma;\delta)}{2}$$

$$V_2^3(\sigma;\delta) = \frac{(1-\delta)u_2(f_1^3(\sigma)) + \delta V_2^2(f_1^3(\sigma);\delta) + (1-\delta)u_2(f_2^3(\sigma)) + \delta V_2^2(f_2^3(\sigma);\delta)}{2}$$

$$= \frac{(1-\delta)u_2(\sigma)+\delta V_2^2(\sigma;\delta) + h_2((1-\delta)u_1(\sigma)+\delta V_1^2(\sigma;\delta))}{2} \tag{37}$$

Computing these in the context of the present example gives

$$V_1^3(\sigma^0;\delta) = \frac{\frac{12-[(1-\delta)u_2(\sigma^0)+\delta V_2^2(\sigma^0;\delta)]}{2} + (1-\delta)u_1(\sigma^0)+\delta V_1^2(\sigma^0;\delta)}{2} = 3.75 + 0.375\delta \tag{38}$$

$$V_2^3(\sigma^0;\delta) = \frac{(1-\delta)u_2(\sigma^0)+\delta V_2^2(\sigma^0;\delta) + \frac{12-[(1-\delta)u_1(\sigma^0)+\delta V_1^2(\sigma^0;\delta))]}{2}}{2} = 3 + 0.5625\delta \tag{39}$$

$$\mathbf{V}^3(\sigma^0;\delta) = (3.75 + 0.375\delta, 3 + 0.5625\delta) \tag{40}$$

Third Period. In the third period, $\mathbf{V}^3(\sigma^0;\delta)$ constitutes the status quo (in payoff terms). It turns out that the mapping h_2 describing efficient boundary changes depending on δ. For $\delta < 2/3$, $h_2(u_1) = \frac{1}{2}(12 - u_1)$; while for $\delta \geqslant 2/3$, $h_2(u_1) = 12 - 2u_1$. By reasoning in a manner similar to that in second period, we compute that for $\delta \geqslant 2/3$,

$$V_1^4(\sigma^0;\delta) = \frac{\frac{12-[(1-\delta)u_2(\sigma^0)+\delta V_2^3(\sigma^0;\delta)]}{2} + (1-\delta)u_1(\sigma^0)+\delta V_1^3(\sigma^0;\delta)}{2} = 3.75 + 0.375\delta + 0.046875\delta^2 \tag{41}$$

$$V_2^4(\sigma^0;\delta) = \frac{(1-\delta)u_2(\sigma^0)+\delta V_2^3(\sigma^0;\delta) + 12 - 2[(1-\delta)u_1(\sigma^0)+\delta V_1^3(\sigma^0;\delta))]}{2} = 4.5 - 0.75\delta - 0.09375\delta^2 \tag{42}$$

$$\mathbf{V}^4(\sigma^0;\delta) = (3.75 + 0.375\delta + 0.046875\delta^2, 4.5 - 0.75\delta - 0.09375\delta^2) \tag{43}$$

For $\delta \geqslant 2/3$, $\mathbf{V}^4(\sigma^0;\delta)$ is the 3-period noncooperative Raiffa solution of the bargaining model for the present example. Note that since $2V_1^4(\sigma^0;\delta) + V_2^4(\sigma^0;\delta) = 12$, $\mathbf{V}^4(\sigma^0;\delta)$ lies on the efficient boundary, and $\lim_{\delta\to 1}\mathbf{V}^4(\sigma^0;\delta) = (4.171875, 3.65625)$.

Letting $\mathbf{R}^k = \lim_{\delta\to 1}\mathbf{V}^k(\sigma^0;\delta)$ for every k, we get the Raiffa negotiation curve as the piece-wise linear curve formed by joining the payoff points in the finite Raiffa sequence, given by

$$\mathbf{R}^1 = (2,1), \quad \mathbf{R}^2 = (3.75, 3), \quad \mathbf{R}^3 = (4.125, 3.5625), \quad \mathbf{R}^4 = (4.171875, 3.65625)$$

This Raiffa negotiation curve is shown in Figure 3 in section 4. Note that Raiffa's bargaining solution $\mathbf{R}^4$ for this example reflects the asymmetry in the initial status quo, while the Nash bargaining solution does not.

The equilibrium path from σ^0 will feature contracts proposed every period by players randomly chosen as proposers. The solution $\mathbf{V}^4(\sigma^0;\delta)$ is the average of continuation values of these paths starting from σ^0 in period 1. In this example, the three steps of the Raiffa negotiation curve appear as a sequence of recursively computed threat points $\lim_{\delta\to1}\mathbf{V}^1(\sigma^0;\delta)$, $\lim_{\delta\to1}\mathbf{V}^2(\sigma^0;\delta)$, and $\lim_{\delta\to1}\mathbf{V}^3(\sigma^0;\delta)$, leading to Raiffa's solution $\mathbf{R}^4=\lim_{\delta\to1}\mathbf{V}^4(\sigma^0;\delta)$ as the equilibrium bargaining value.

Example 2. Suppose we have the same $\mathcal{F}$ as in Example 1; but the initial contractual state σ^0 is such that $\mathbf{u}(\sigma^0)=(1,1)$. For this bargaining problem $(\mathcal{F},\mathbf{u}(\sigma^0))$, the Nash bargaining solution and the Kalai Smorodinsky solution coincide and is the point $(4,4)$ on the efficient boundary. Using the same reasoning as in Example 1, we compute

$$\begin{aligned}
\mathbf{V}^1(\sigma^0;\delta) &= (1,1)\\
\mathbf{V}^2(\sigma^0;\delta) &= (3.25,3.25)\\
\mathbf{V}^3(\sigma^0;\delta) &= (3.25+0.5625\delta,3.25+0.5625\delta)\\
\mathbf{V}^4(\sigma^0;\delta) &= (3.25+0.5625\delta+0.140625\delta^2,3.25+0.5625\delta+0.140625\delta^2)\\
&\quad\text{and so on ...}
\end{aligned}$$

In this example, all the k- period noncooperative Raiffa solutions lie on the same line segment joining the initial status quo $(1,1)$ to the extreme point $(4,4)$ on the efficient boundary. Moreover, unlike Example 1, for no finite k, does $\mathbf{V}^k(\sigma^0;\delta)$ lie on the efficient boundary. It is also clear that $\lim_{\delta\to1}\mathbf{V}^k(\sigma^0;\delta)=(4,4)$, which is also the Nash bargaining solution and the Kalai Smorodinsky solution for the bargaining problem.

The Raiffa negotiation curve is a straight line joining the payoff points in the infinite Raiffa sequence, whose first four points are given by

$$\mathbf{R}^1=(1,1),\quad \mathbf{R}^2=(3.25,3.25),\quad \mathbf{R}^3=(3.8125,3.8125),\quad \mathbf{R}^4=(3.953125,3.953125)$$

This linear Raiffa negotiation curve comprising of infinite steps for this example is shown in Figure 4 in section 4.

Theorem 2. *(Convergence of Noncooperative Raiffa Solution to Nash Bargaining Solution). As the deadline frictions and then the discounting frictions disappear, the noncooperative Raiffa solution with respect to the bargaining model $\mathscr{E}(G^k,\sigma^0,\delta)$ converges to the Nash bargaining solution of the bargaining problem $(\mathcal{F},\mathbf{u}(\sigma^0))$. Formally,* $\lim_{\delta\to1}\lim_{k\to\infty}\mathbf{V}^{k+1}(\sigma^0;\delta)=\mathbf{N}(\mathcal{F},\sigma^0)$.

Proof. As $k \to \infty$, Proposition 3 in the bargaining model $\mathscr{E}(G^k, \sigma^0, \delta)$ coincides with Proposition 1 in the bargaining model $\mathscr{E}(G^\infty, \sigma^0, \delta)$. The convergence claim then follows from the corresponding convergence (as $\delta \to 1$) result, Proposition 2, in $\mathscr{E}(G^\infty, \sigma^0, \delta)$. Q.E.D.

Our definition of the noncooperative Raiffa solution and the convergence result - Theorem 2, help clarify the remarks in Luce and Raiffa (1957) on pp.136-137. Geometrically, we may think of the sequence in k of the k-period noncooperative Raiffa solutions starting from the initial state $\mathbf{u}(\sigma^0)$ as the possibly nonlinear *negotiation curve* in the feasible payoff space $\mathcal{F}$. In the bargaining model $\mathscr{E}(G^k, \sigma^0, \delta)$ with deadline and discounting frictions, players care about the payoffs they incur at every step in this negotiation curve. By Theorem 2, as the frictions disappear, players only care about the final contractual state i.e. where they end up on the efficient boundary. This corresponds to a linear path from $\mathbf{u}(\sigma^0)$ to the final payoff state on the efficient boundary - the Nash bargaining solution of the problem $(\mathcal{F}, \mathbf{u}(\sigma^0))$ which is equal to Raiffa's bargaining solution by Theorem ?THM? **??**. The following continuation of Example 1 illustrates that with frictions, even though the 3-period noncooperative Raiffa solution lies on the efficient boundary, if we add more periods to the model, the movement from 3-period to 4-period noncooperative Raiffa solution is on the efficient boundary. This movement is required for the convergence result of Theorem 2.

Example 1 (continued). Suppose we prefix another period to the bargaining model in Example 1. Then in the *fourth period*[8], for $\delta \geqslant 2/3$,

$$\mathbf{V}^4(\sigma^0; \delta) = (3.75 + 0.375\delta + 0.046875\delta^2, 4.5 - 0.75\delta - 0.09375\delta^2)$$

constitutes the status quo or the threat point for bargaining in the fourth period; and the relevant h_1 and h_2 are given by $h_1(u_2) = \frac{1}{2}(12 - u_1)$ and $h_2(u_1) = 12 - 2u_1$. This leads to the equilibrium bargaining values

$$V_1^5(\sigma^0; \delta) = \frac{\frac{12 - [(1-\delta)u_2(\sigma^0) + \delta V_2^4(\sigma^0;\delta)]}{2} + (1-\delta)u_1(\sigma^0) + \delta V_1^4(\sigma^0; \delta)}{2} = 3.75 + 0.375\delta^2 + 0.046875\delta^3 \tag{44}$$

$$V_2^5(\sigma^0; \delta) = \frac{(1-\delta)u_2(\sigma^0) + \delta V_2^4(\sigma^0; \delta) + 12 - 2[(1-\delta)u_1(\sigma^0) + \delta V_1^4(\sigma^0; \delta))]}{2} = 4.5 - 0.75\delta^2 - 0.09375\delta^3 \tag{45}$$

$$\mathbf{V}^5(\sigma^0; \delta) = (3.75 + 0.375\delta^2 + 0.046875\delta^3, 4.5 - 0.75\delta^2 - 0.09375\delta^3) \tag{46}$$

Note that $2V_1^5(\sigma^0; \delta) + V_2^5(\sigma^0; \delta) = 12$, so $\mathbf{V}^5(\sigma^0; \delta)$ is a point on the efficient boundary.

[8]this is the last period with respect to backward counting of periods but first period with respect to forward counting.

Moreover, $V_1^5(\sigma^0;\delta) < V_1^4(\sigma^0;\delta)$ and $V_2^5(\sigma^0;\delta) > V_2^4(\sigma^0;\delta)$. So the movement from $\mathbf{V}^4(\sigma^0;\delta)$ to $\mathbf{V}^5(\sigma^0;\delta)$ is along the efficient boundary. On adding further periods, such movement will keep repeating, converging to the Nash solution $(4,4)$ in the limit.

8 Discussion

Raiffa (1953) came up with a solution to the bargaining problem independently of Nash, but, unlike Nash, he did not provide an axiomatic foundation for his solution. However, he did consider his work to be on arbitration schemes, while Nash, as later expounded by Roth (1979), was using his axioms to model features of positive bargaining theory. The "negotiation curve" in Raiffa's paper does have a non-cooperative flavor, since the negotiation curve evolves over time according to the actions of individual bargainers. One could interpret it, as various authors have (from Kalai (1977) to Hu and Rocheteau (2020)) as step-by-step negotiation, in which the entire surplus from negotiation is not available immediately for an agreement to be reached. Myerson (1991) relies on the short description in Luce and Raiffa (1957) to define the Raiffa solution. He constructs a sequential bargaining, finite horizon model, which implements the Raiffa solution in the limit as the deadline frictions disappear. However, the unique subgame perfect equilibrium in his model leads to an immediate solution, so no negotiation curve appears in equilibrium.

Our exploration of Raiffa bargaining solution also uses the short description in Luce and Raiffa (1957), where the authors speak of linearization of the negotiation curve. It seems to us that the nonlinear negotiation curve arises from some friction in the model and what Luce and Raiffa call linearization is a limiting result as these frictions disappear. We also feel that the insights from step-by-step negotiation models - that the contract from one period serves as the status quo for the next period - are crucial to the Raiffa solution. We therefore model a strategic-form game being played every period and contractual arrangements being proposed and possibly accepted of one-period duration. The frictions are the finite number of steps with the status quo for one step being derived from the agreement in a previous step, as well as the usual discounting friction. As both these frictions disappear, individual period status quo points become unimportant, as does delay, and the solution converges to what Luce and Raiffa describe as a linearization of the status quo point, or the Nash bargaining solution.

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Appendices

In these appendices, we present two bargaining models in the spirit of Binmore et al. (1986) and Rubinstein (1982) for negotiating joint (correlated) strategies in an underlying finite one-shot strategic game, with an assumption about the way the status quo point relates to Pareto frontier of the game polytope $\mathcal{F}$. Although the efficient boundary of the strategic game may be non-differentiable at finite number of points, the equilibrium characterization as well as the uniqueness and convergence properties of equilibrium can be established using arguments similar to Shaked and Sutton (1984), Binmore and Dasgupta (1987), and Binmore et al. (1986). For this reason, we simply state[9] the propositions for the two models.

Assumption 3 in the risk-of-breakdown model, and Assumption 4 in the time preference model, are requirements on the way the breakdown point $\mathbf{v}$ and the deadlock point $\mathbf{0}$ relate to $\mathcal{F}$. Specifically, they require that the breakdown point and the deadlock point are not worse for any player than her most unfavorable point on the efficient boundary of $\mathcal{F}$. Put another way, for both players, there is always some point on the efficient boundary that is no better than their breakdown/deadlock point. This is typical of most bargaining situations[10].

A A bargaining model with risk-of-breakdown

Let $\sigma \in \Delta(\mathcal{A})$ denotes a correlated action. While bargaining, players can write a binding contract to jointly coordinate over and play a correlated action. Contracts are negotiated through a multi-period bargaining game $\mathscr{E}_{rb}(G, \delta)$; and once settled, are assumed to be enforced at the time the game G is played. The set of times T at which offers can be made is

[9] The proofs are available from the authors on request.

[10] For instance, for two countries who have an opportunity to negotiate a boundary dispute, the most unfavorable point on the efficient boundary would be to legally cede control of the disputed territory. This is worse, or at least no better than, the deadlock point even when the disputed territory is *de-facto* controlled by one country.

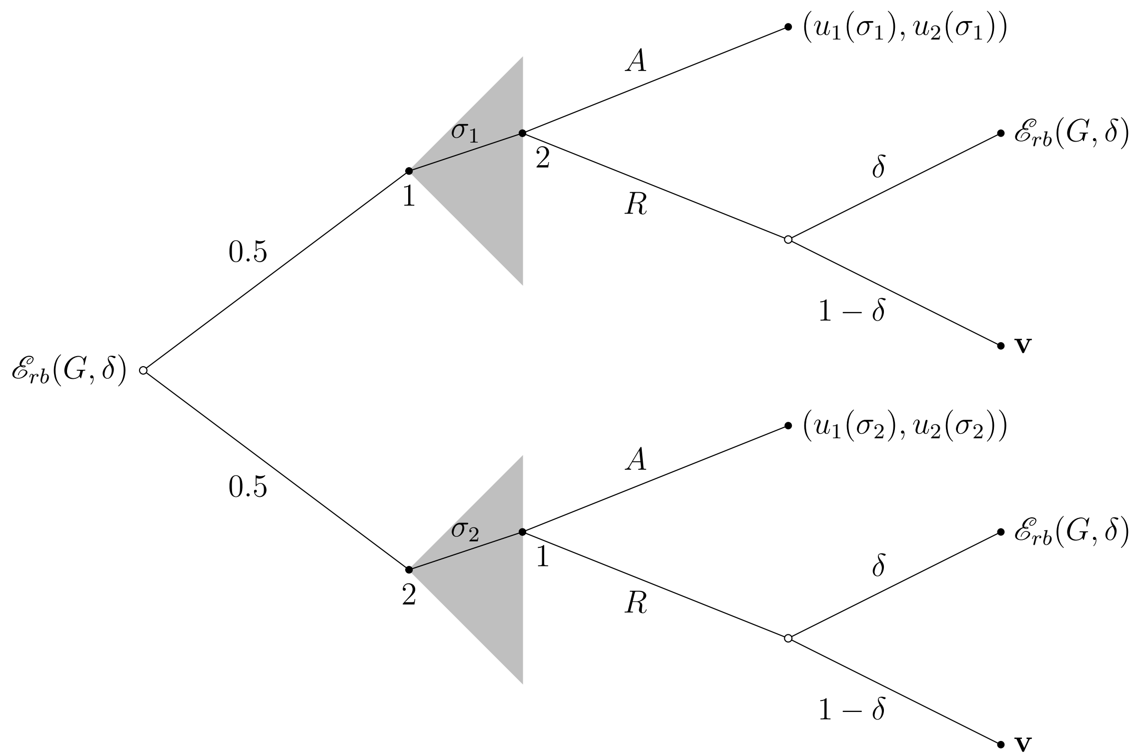

Figure 7: Recursive Schema of Extensive Form Bargaining Game $\mathscr{E}_{rb}(G,\delta)$

discrete, starts at $t = 1$ and continues until bargaining ends. Period t refers to the duration of time between time t and time $t+1$. We describe the bargaining protocol recursively. Figure 7 gives a schematic extensive form description.

Suppose at time t, players have not agreed to any contract and bargaining has not ended yet. A player $i \in N$ is randomly chosen with probability $1/2$ to be a proposer. A contractual offer is a correlated action $\sigma \in \Delta(\mathcal{A})$, and requires acceptance by the other player - the responder, to become a binding contract. Therefore, an accept (A) or reject (R) decision from the responder is solicited.

Once an offer σ becomes a binding contract in period t, bargaining ends. However, if the offer is rejected in period t, then bargaining ends with probability $1-\delta$. With probability $0 < \delta < 1$, bargaining continues to period $t+1$.

When bargaining ends with an agreement on the correlated strategy contract σ, then the strategic form game G is played with all players implementing σ by means of a joint randomization device. On the other hand, when bargaining ends without an agreement, then we say a breakdown has occured.

Payoffs. When bargaining ends with an agreement on the correlated strategy contract σ, players receive the payoffs $\mathbf{u}(\mathbf{a}) = (u_1(\mathbf{a}), u_2(\mathbf{a}))$ with probability $\sigma(\mathbf{a})$. When bargaining ends in a breakdown, players get the payoffs as specified by an exogenous breakdown point $\mathbf{v} = (v_1, v_2) \in \mathcal{F}$.

Strategies. A stationary bargaining strategy b_i for player i is a pair (σ_i, r_i) of offer and response strategies. i's offer strategy σ_i is a correlated strategy contract in $\Delta(\mathcal{A})$. i's response strategy r_i must accept or reject any offer directed towards her, and is therefore, a function $r_i : \Delta(\mathcal{A}) \mapsto \{A, R\}$.

Solution Concept. A profile of stationary bargaining strategies $\mathbf{b} = (b_1, b_2)$ is a subgame perfect equilibrium (SSPE) of the bargaining protocol if they constitute a Nash equilibrium after every history of play.

We now introduce an assumption on the location of the breakdown point of the bargaining model in relation to the underlying strategic game. We will derive the uniqueness and convergence results in the model under this assumption.

Assumption 3. *The breakdown point* $\mathbf{v}$ *satisfies* $v_i \geqslant m_i(\partial\mathcal{F})$ *for every player* $i = 1, 2$.

Since players are vNM expected utility maximizers, we have $u(\Delta(\mathcal{A})) = \mathcal{F}$. In words, the correlated strategy payoffs span the convex hull of the pure strategy payoffs in the game. Therefore, any contractual offer of a correlated strategy $\sigma \in \Delta(\mathcal{A})$ may be equivalently viewed as a payoff offer $\mathbf{u}(\sigma) \in \mathcal{F}$. Let $\mathbf{V}(\mathbf{b}; \delta) = (V_1(\mathbf{b}; \delta), V_2(\mathbf{b}; \delta))$ be the values of strategy profile $\mathbf{b}$ in the bargaining game $\mathscr{E}_{rb}(G, \delta)$. Using dynamic programming, we characterize the win-win stationary bargaining equilibria and the corresponding values as

Proposition A.1. *(SSPE). Suppose G satisfies Assumption 3. A profile of stationary bargaining strategies $\mathbf{b} = (b_1, b_2)$ where $b_i = (\sigma_i, r_i)$ for $i = 1, 2$ constitutes a subgame-perfect equilibrium of the bargaining game $\mathscr{E}_{rb}(G, \delta)$ if and only if*

(i) (Strategy Optimality). Given the associated pair of values $\mathbf{V}(\mathbf{b}; \delta) = (V_1(\mathbf{b}; \delta), V_2(\mathbf{b}; \delta))$,
(a) (Response Optimality). For every player $i = 1, 2$, $r_i(\sigma) = A$ if and only if $u_i(\sigma) \geqslant (1-\delta)v_i + \delta V_i(\mathbf{b}; \delta)$
(b) (Offer Optimality). Player 1's offer $\sigma_1 \in \Delta(\mathcal{A})$ and player 2's offer $\sigma_2 \in \Delta(\mathcal{A})$ are such that

$$\mathbf{u}(\sigma_1) = \Big(h_1\big((1-\delta)v_2 + \delta V_2(\mathbf{b}; \delta)\big), (1-\delta)v_2 + \delta V_2(\mathbf{b}; \delta)\Big) \tag{47}$$

$$\mathbf{u}(\sigma_2) = \Big((1-\delta)v_1 + \delta V_1(\mathbf{b}; \delta), h_2\big((1-\delta)v_1 + \delta V_1(\mathbf{b}; \delta)\big)\Big) \tag{48}$$

(ii) (Value Optimality). The associated pair of values $\mathbf{V}(\mathbf{b}; \delta) = (V_1(\mathbf{b}; \delta), V_2(\mathbf{b}; \delta))$ satisfy

the Bellman equations of optimality given by

$$V_1(\delta) = \frac{h_1\big((1-\delta)v_2 + \delta V_2(\delta)\big) + (1-\delta)v_1 + \delta V_1(\delta)}{2} \tag{49}$$

$$V_2(\delta) = \frac{(1-\delta)v_2 + \delta V_2(\delta) + h_2\big((1-\delta)v_1 + \delta V_1(\delta)\big)}{2} \tag{50}$$

Proposition A.2. *(Uniqueness of SSPE Values). Suppose G satisfies Assumption* 3. *For any $\delta \in [0,1)$, the Bellman equations of optimality for stationary subgame perfect equilibrium strategies are given by* (49) *and* (50)*, and they have a unique solution $\mathbf{V}(\delta) = (V_1(\delta), V_2(\delta))$ that satisfies $V_1(\delta) \geqslant v_1$ and $V_2(\delta) \geqslant v_2$.*

Proposition A.3. *(Uniqueness of SPE Values). Suppose G satisfies Assumption* 3. *For any $\delta \in [0,1)$, the bargaining game $\mathscr{E}(G,\delta)$ has a unique subgame perfect equilibrium payoff $\mathbf{V}(\delta) = (V_1(\delta), V_2(\delta))$.*

Proposition A.4. *(Convergence to Nash Bargaining Solution). Suppose G satisfies Assumption* 3. *As $\delta \to 1$, the sequence in δ, of the unique SPE values $\mathbf{V}(\delta)$ of the bargaining game $\mathscr{E}_{rb}(G,\delta)$ converges to the Nash bargaining solution of $(\mathcal{F}, \mathbf{v})$.*

B A bargaining model with time preference

In this section, we consider a variant of the bargaining model of Section A, where there is no contingency of a breakdown. However, players have an option of never playing the game in question and they have a preference over reaching an agreement sooner than later. Let $\delta \in (0,1)$ be the common discount factor that players use to discount future payoffs. Using the terminology of Binmore (1994), we refer to the disagreement corresponding to the outcome in which no agreement is reached at any finite date as a *deadlock*, denoted by d. With discounted preferences, the *deadlock point*, defined as the value of deadlock, is $\mathbf{0} = (0,0)$.

We suppose that players have an opportunity to play a strategic game G given in preference form, $\langle N, (\mathcal{A}_i, \succsim_i)_{i\in N}\rangle$, where $\succsim_i$ is the preference relation of player i on the set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ of action profiles. We derive the relevant $\mathcal{F}$ for the bargaining model with time preference by adding a distinguished action d_i to the action set $\mathcal{A}_i$ for every player $i = 1,2$; specifying that each player has the option of unilaterally enforcing the deadlock outcome d by choosing d_i; extending the preference relation $\succsim_i$ to the set of lotteries over $\mathcal{A} \cup \{d\}$; postulating that the extended preference relations satisfy the vNM axioms so that they can be represented by the expectation of vNM utility functions $u_i : \mathcal{A} \cup \{d\} \mapsto \mathbb{R}$ and, finally calibrating

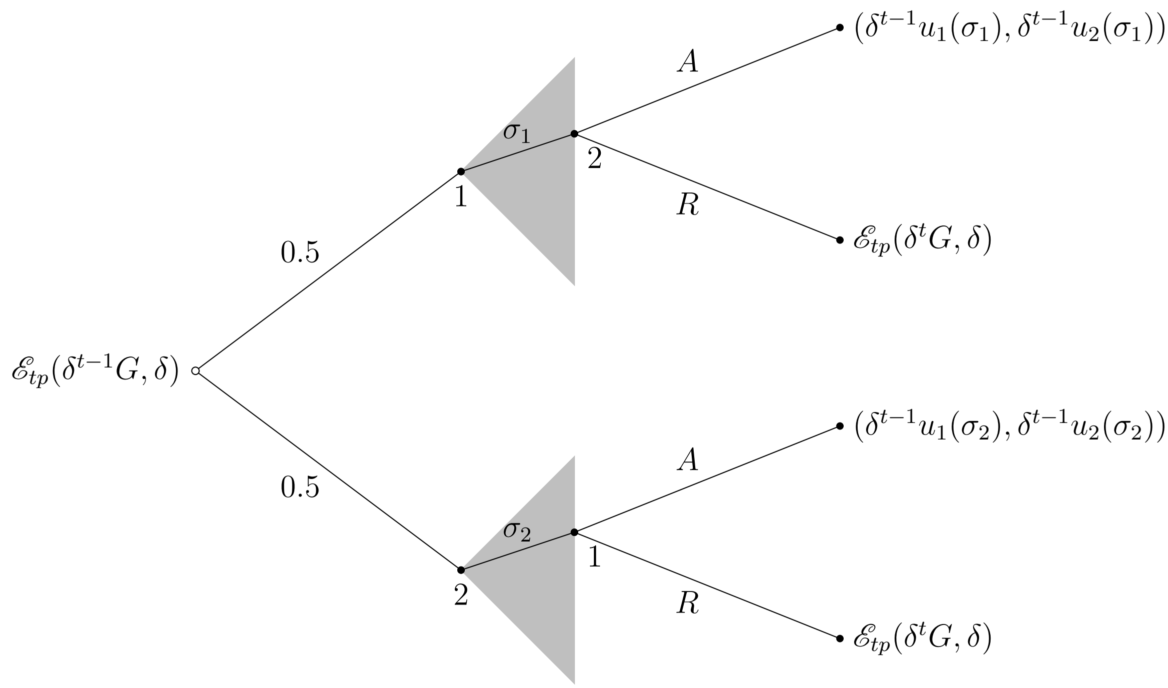

Figure 8: Recursive Schema of Extensive Form Bargaining Game $\mathscr{E}_{tp}(G, \delta)$. For any date t, $\delta^t G$ is the strategic game $\langle N, \mathcal{A}, \delta^t \mathbf{u} \rangle$.

both players' vNM utility functions u_i to assign the value 0 to the deadlock outcome d. We refer to the augmented strategic game in utility form so obtained as $\hat{G}$. The feasible set $\mathcal{F}$ of payoffs for the bargaining model with time preference is then defined to be the convex hull of pure-action payoff profiles in $\hat{G}$. This ensures that the deadlock point $\mathbf{0} \in \mathcal{F}$. Even though risk preferences are fundamental in the definition of a strategic game, we refer to the present formulation as the bargaining model with time preference and denote it by $\mathscr{E}_{tp}(G, \delta)$. This is because time preferences are the only source of bargaining frictions in this model. The model is basically the well known Rubinstein (1982) model, but with the payoff environment described by a strategic form game. Figure 8 gives a schematic extensive form description.

We introduce an assumption which is needed in the uniqueness proposition for the time preference model.

Assumption 4. *The deadlock point* $\mathbf{0}$ *satisfies* $0 \geqslant m_i(\partial \mathcal{F})$ *for every player* $i = 1, 2$.

Assumption 4 is an assumption on the way the deadlock point relates to the efficient boundary of the set of contractually feasible payoffs in the underlying game.

Proposition $B.1$ describes the stationary bargaining equilibria in the time preference model in terms of strategy optimality and value optimality in a manner that parallels Proposition

A.1 for the risk of breakdown model.

Proposition B.1. *(SSPE). Suppose G satisfies Assumption 4. A profile of stationary bargaining strategies $\mathbf{b} = (b_1, b_2)$ where $b_i = (\sigma_i, r_i)$ for $i = 1, 2$ constitutes a subgame-perfect equilibrium of the bargaining game $\mathscr{E}_{tp}(G, \delta)$ if and only if*

(i) (Strategy Optimality). Given the associated pair of values $\mathbf{V}(\mathbf{b}; \delta) = (V_1(\mathbf{b}; \delta), V_2(\mathbf{b}; \delta))$,
(a) (Response Optimality). For every player $i = 1, 2$, $r_i(\sigma) = A$ if and only if $u_i(\sigma) \geqslant \delta V_i(\mathbf{b}; \delta)$
(b) (Offer Optimality). Player 1's offer $\sigma_1 \in \Delta(\mathcal{A})$ and player 2's offer $\sigma_2 \in \Delta(\mathcal{A})$ are such that

$$\mathbf{u}(\sigma_1) = \Big(h_1\big(\delta V_2(\mathbf{b}; \delta)\big), \delta V_2(\mathbf{b}; \delta)\Big) \tag{51}$$

$$\mathbf{u}(\sigma_2) = \Big(\delta V_1(\mathbf{b}; \delta), h_2\big(\delta V_1(\mathbf{b}; \delta)\big)\Big) \tag{52}$$

(ii) (Value Optimality). The associated pair of values $\mathbf{V}(\mathbf{b}; \delta) = (V_1(\mathbf{b}; \delta), V_2(\mathbf{b}; \delta))$ satisfy the Bellman equations of optimality given by

$$V_1(\delta) = \frac{h_1\big(\delta V_2(\delta)\big) + \delta V_1(\delta)}{2} \tag{53}$$

$$V_2(\delta) = \frac{\delta V_2(\delta) + h_2\big(\delta V_1(\delta)\big)}{2} \tag{54}$$

Proposition B.2. *(Uniqueness of SSPE Values). Suppose G satisfies Assumption 4. For any $\delta \in [0, 1)$, the Bellman equations of optimality for stationary subgame perfect equilibrium strategies are given by* (53) *and* (54)*, and they have a unique solution $\mathbf{V}(\delta) = (V_1(\delta), V_2(\delta))$.*

Proposition B.3. *(Uniqueness of SPE Values). Suppose G satisfies Assumption 4. For any $\delta \in [0, 1)$, the bargaining game $\mathscr{E}_{tp}(G, \delta)$ has a unique subgame perfect equilibrium payoff $\mathbf{V}(\delta) = (V_1(\delta), V_2(\delta))$.*

Proposition B.4. *(Convergence to Nash Bargaining Solution). Suppose G satisfies Assumption 4. As $\delta \to 1$, the sequence in δ, of the unique SPE values $\mathbf{V}(\delta)$ of the bargaining game $\mathscr{E}_{tp}(G, \delta)$ converges to the Nash Bargaining Solution of $(\mathcal{F}, \mathbf{0})$.*

C On Raiffa's mediation interpretation

Let us modify the bargaining model $\mathscr{E}(G^{\infty}, \sigma^0, \delta)$ by adding a mediator who is not a player in the game. The sequence of events in period t is as follows. At the start, the model is in some state σ. At this point, the mediator offers a contract $m(\sigma) \in \Delta(\mathcal{A})$ to players, which they then accept or reject in some order. If the mediated contract $m(\sigma)$ is rejected by some player, then the rest of the period is exactly as in $\mathscr{E}(G^{\infty}, \sigma^0, \delta)$ - players bargain in random proposer fashion to determine the end-of-period state σ', collect their payoffs in state σ' and the model moves to next date $t+1$ in state σ'. If the mediated contract $m(\sigma)$ is accepted by both players, then the model state changes this period to $m(\sigma)$, there is no strategic bargaining this period, players collect their payoffs in state $m(\sigma)$, and the model moves to next date $t+1$ in state $m(\sigma)$. Call the mediator augmented bargaining model as $\mathscr{E}^m(G^{\infty}, \sigma^0, \delta)$.

A stationary Markov mediation strategy is a function $m : \Delta(\mathcal{A}) \mapsto \Delta(\mathcal{A})$, which for any state σ, gives the mediated contract $m(\sigma)$ that is proposed by the mediator. A response strategy $rm_i : (\sigma, m(\sigma)) \mapsto \{A, R\}$ of player i is a rule which describes whether to accept or reject the mediated offer $m(\sigma)$ in state σ. A profile of such response strategies are denoted by $\mathbf{rm} = (rm_1, rm_2)$. A strategy profile in the model, which is a vector $\beta = (m, \mathbf{rm}, \mathbf{b})$ consisting of mediation strategy of the mediator, response-to-mediator strategies of players, and bargaining strategies of the players, describes a complete path of play. Let $\mathbf{V}(\bullet, \beta; \delta) : \Delta(\mathcal{A}) \mapsto \mathcal{F}$ denote the value function of the strategy profile β. In every state σ, $\mathbf{V}(\sigma, \beta; \delta) = (V_1(\sigma, \beta; \delta), V_2(\sigma, \beta; \delta))$ is the pair of values for players. A strategy profile has a *mediated* outcome path if the contract proposed by the mediator is accepted by both players in every state.

Suppose $\mathbf{V}(\bullet; \delta) : \Delta(\mathcal{A}) \mapsto \mathcal{F}$ is the unique (by Proposition 2) value function shared by all SMPE of $\mathscr{E}(G^{\infty}, \sigma^0, \delta)$. Consider the following strategy of the mediator

$$\text{For every } \sigma, \quad \text{choose } m(\sigma) \text{ such that } \mathbf{u}(m(\sigma)) = \mathbf{V}(\sigma; \delta) \tag{55}$$

In other words, $m(\sigma)$ is the correlated strategy contract whose vNM payoff to players is equal to their unique SMPE payoff in state σ in the model $\mathscr{E}(G^{\infty}, \sigma^0, \delta)$. For $t \geqslant 2$, inductively define $m^t(\sigma) := m(m^{t-1}(\sigma))$. Then the value of the mediated outcome path in state σ, assuming the mediator follows the strategy defined in (55) and both players accept the mediator's offer $m(\sigma)$ in every state σ is given by

$$\mathbf{V}^m(\sigma; \delta) = (1-\delta)[\mathbf{u}(m(\sigma)) + \delta \mathbf{u}(m^2(\sigma)) + \delta^2 \mathbf{u}(m^3(\sigma)) + \ldots] \tag{56}$$

Lemma 11. *Suppose G satisfies Assumption 2. Then for any state σ, we have $\mathbf{V}^m(\sigma;\delta) \geqslant \mathbf{V}(\sigma;\delta)$*

Proof. The conclusion of Lemma 11 follows from (56) and the claim: For every $t \geqslant 1$, $\mathbf{u}(m^t(\sigma)) \geqslant \mathbf{V}(\sigma;\delta)$. We prove the claim by mathematical induction. The base step $\mathbf{u}(m(\sigma)) \geqslant \mathbf{V}(\sigma;\delta)$ is immediate from the definition (55) of $m(\sigma)$. For the inductive step, assume the claim is true for every $k = 1,\ldots,t-1$. Then

$$\begin{aligned}
\mathbf{u}(m^t(\sigma)) &= \mathbf{V}(m^{t-1}(\sigma);\delta) && \text{(By definition of } m^t(\sigma)\text{)} && (57)\\
&\geqslant \mathbf{u}(m^{t-1}(\sigma)) && \text{(By Proposition } A.2\text{)} && (58)\\
&\geqslant \mathbf{V}(\sigma;\delta) && \text{(By Inductive Hypothesis)} && (59)
\end{aligned}$$

Q.E.D.

Corollary 3. *Suppose G satisfies Assumption 2. Then for any state σ, the mediator's offer $m(\sigma)$ specified in (55), is accepted by both the players.*

Proof. Lemma 11 implies that for any state σ, the value of the mediator's offer is no less than the value of strategic bargaining for any player. Hence both players accept the mediator's offer. Q.E.D.

Proposition C.1. *Suppose G satisfies Assumption 2. Then for any state σ, the sequence of payoffs $\{\mathbf{u}(m^t(\sigma))\}_{t\geqslant 1}$ along the mediated outcome path is a nondecreasing sequence. Moreover, $\lim_{t\to\infty}\mathbf{u}(m^t(\sigma)) = \mathbf{N}(\mathcal{F},\sigma)$.*

Proof. The conclusion that $\{\mathbf{u}(m^t(\sigma))\}_{t\geqslant 1}$ is a nondecreasing sequence follows from (57) and (58). We also have

$$\begin{aligned}
\lim_{t\to\infty}\mathbf{u}(m^t(\sigma)) &= \lim_{\delta\to 1}\mathbf{V}^m(\sigma;\delta) \\
&\geqslant \lim_{\delta\to 1}\mathbf{V}(\sigma;\delta) && \text{(By Lemma 11)}\\
&= \mathbf{N}(\mathcal{F},\sigma) && \text{(By Proposition 2)}
\end{aligned}$$

Since $\mathbf{N}(\mathcal{F},\sigma)$ is a point on the efficient boundary, the inequality holds as an equality. This proves the proposition. Q.E.D.

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